Motifs for processes on networks

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Outline

1. Motivation and overview
2. Pipeline
3. An example: Covariance and correlation for the OU process
    1. Why care about this example?
    2. OU process
    3. Process motifs
    4. Structure motifs
4. Conclusions

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- What is a “first-order” interaction?
  - One ($m = 1$) interaction between two ($n = 2$) entities

- Higher n

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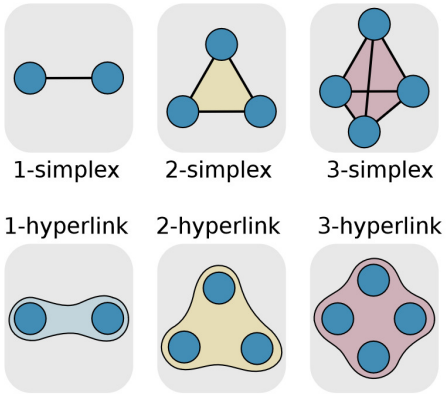

Battiston et al. 2020

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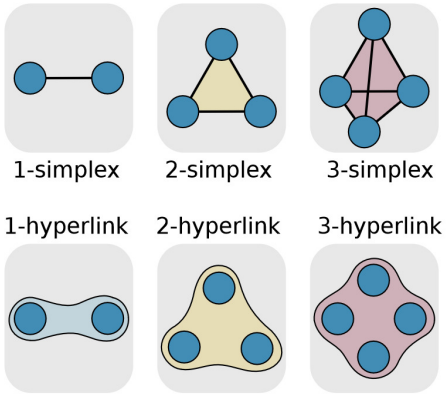
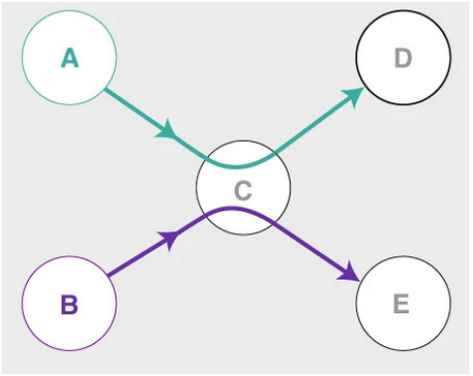


Battiston et al. 2020

- Higher m
  - Walks
  - Paths
  - Motifs



Lambiotte et al. 2019

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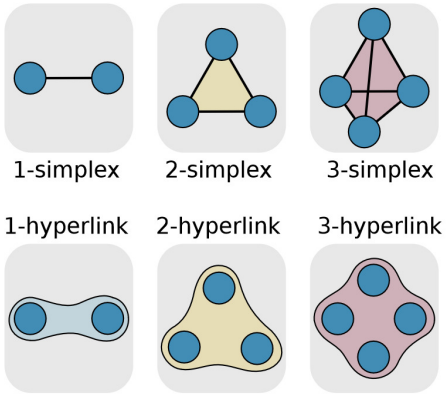

Battiston et al. 2020

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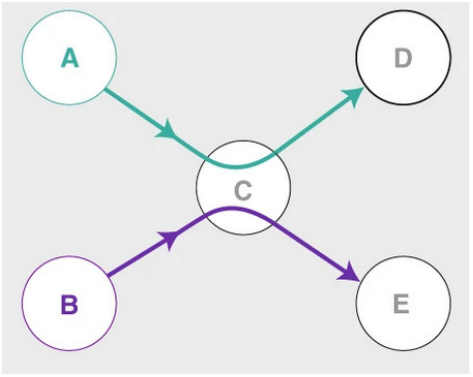
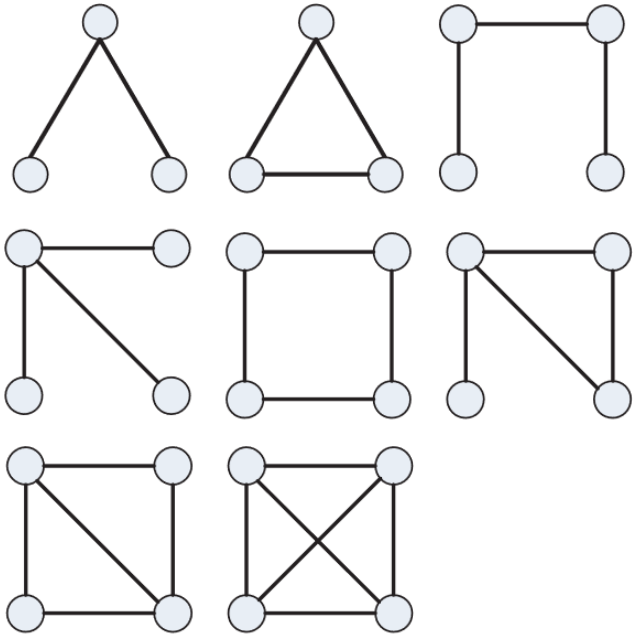


Lambiotte et al. 2019

Motifs in networks

- What is a structural motif?
  - A small, connected subgraph that is **important** for a network's **function**

- What can you do with structural motifs?



Wang et al. 2014

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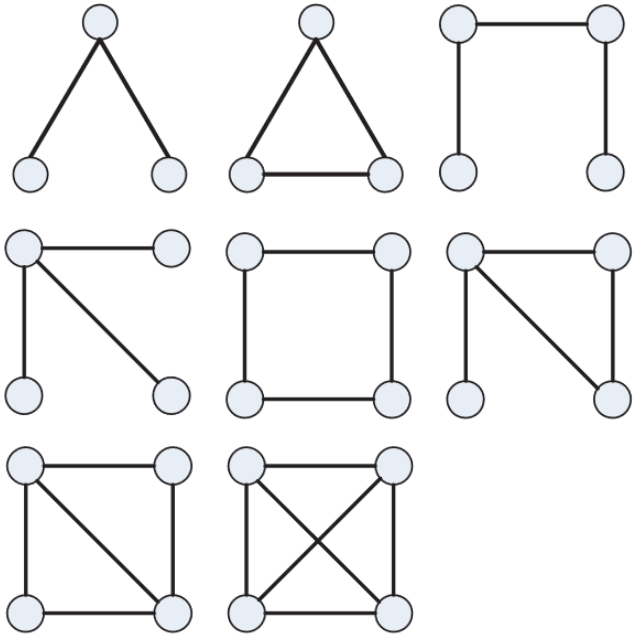

Wang et al. 2014

Motifs in networks

- What is a structural motif?
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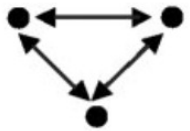
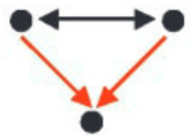
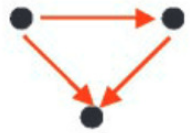
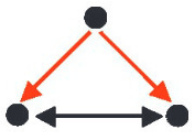
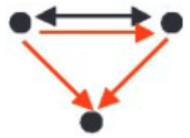
- What can you do with structural motifs?
  - Count them

Illustration†	*N* real	*N* rand ± SD	*z* score
	1,293	14 ± 3.8	332.7
	243	2.4 ± 2.1	115.9
	83	26 ± 6	9.5
	66	2 ± 1.4	46.5
	46	2.7 ± 1.6	26.3

Yeger-Lotem et al. 2004

Motifs in networks

- What is a structural motif?
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- What can you do with structural motifs?
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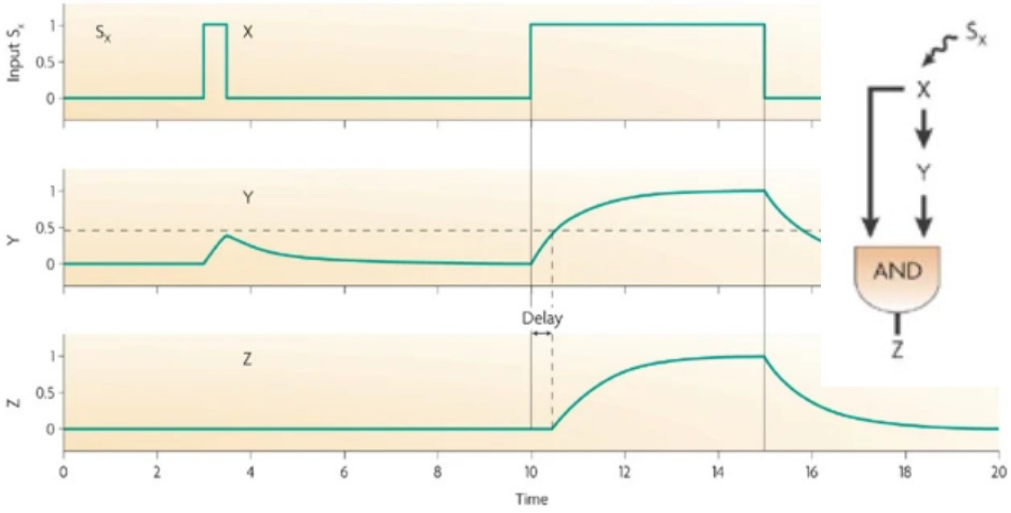
  - Simulate dynamics on **isolated** structural motifs



Alon 2007

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How can we identify motifs that are important for emergent properties of networks?

Pipeline

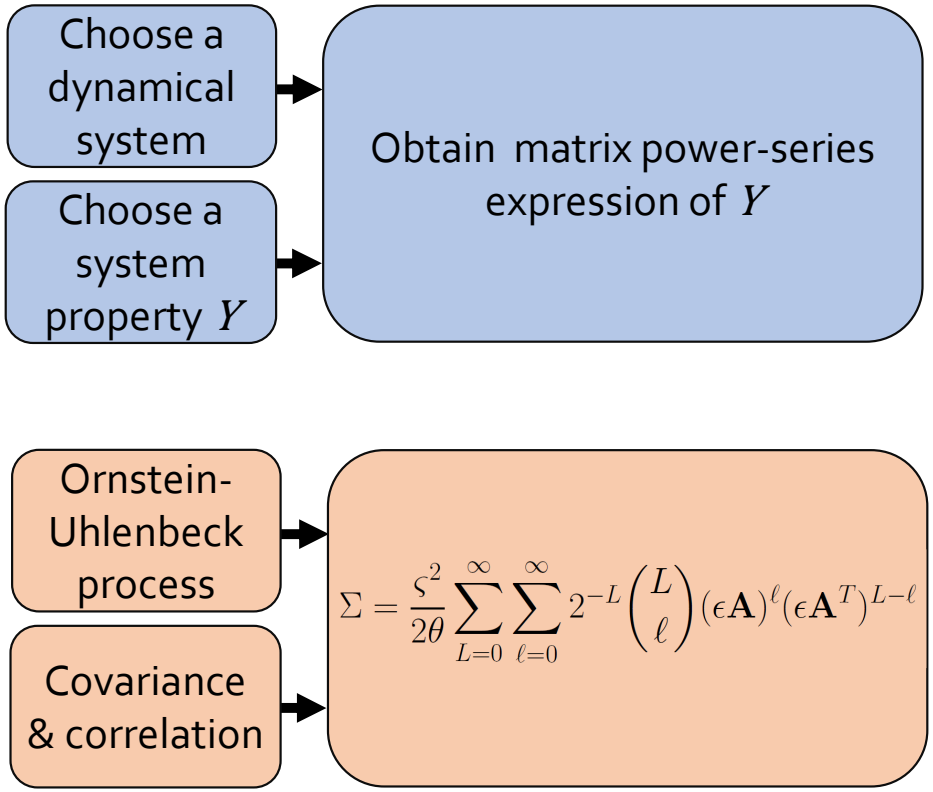
Choose a dynamical system

Choose a system property Y

Ornstein-Uhlenbeck process

Covariance & correlation

Pipeline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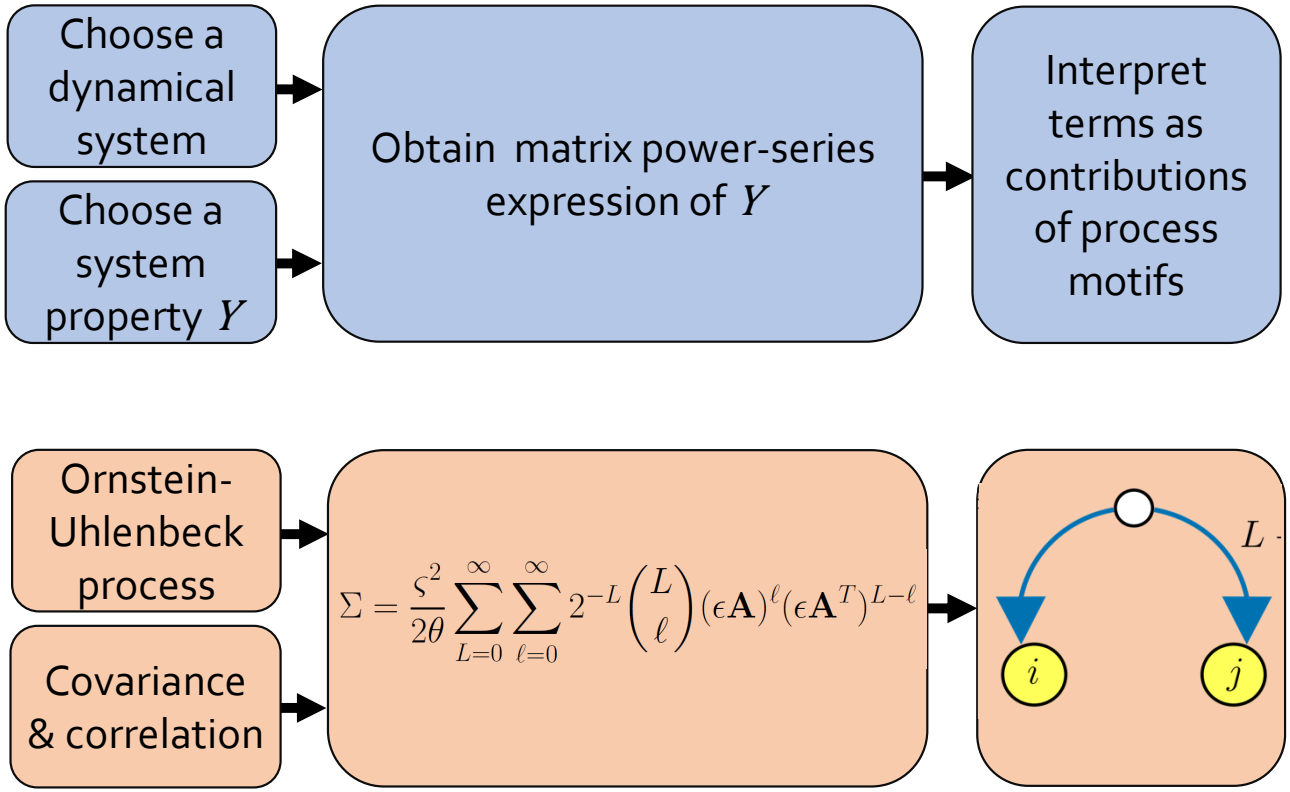
Covariance & correlation
$\Sigma = \frac{\varsigma^2}{2\theta} \sum_{L=0}^{\infty} \sum_{\ell=0}^{\infty} 2^{-L} \binom{L}{\ell} (\epsilon \mathbf{A})^{\ell} (\epsilon \mathbf{A}^T)^{L-\ell}$

Pipeline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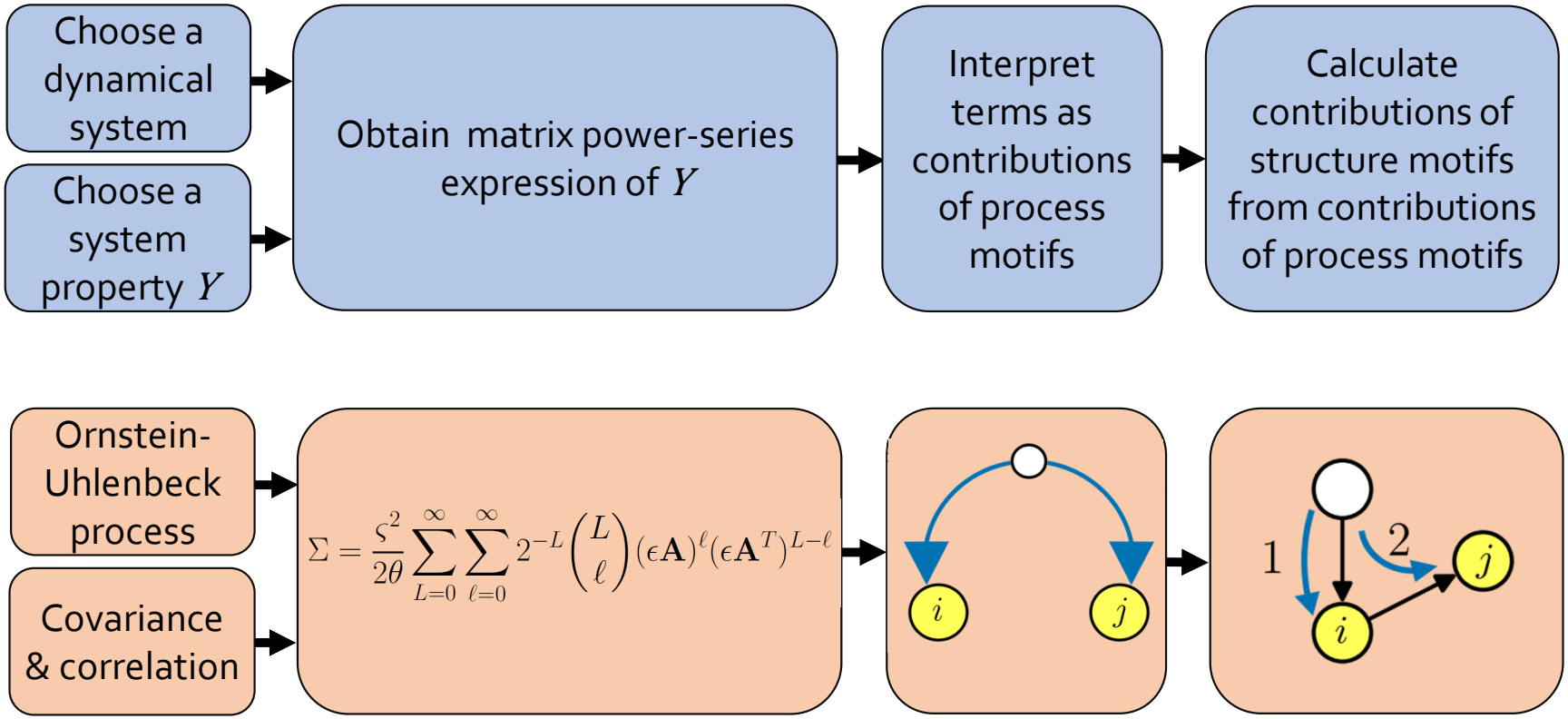
L
i
j

Pipeline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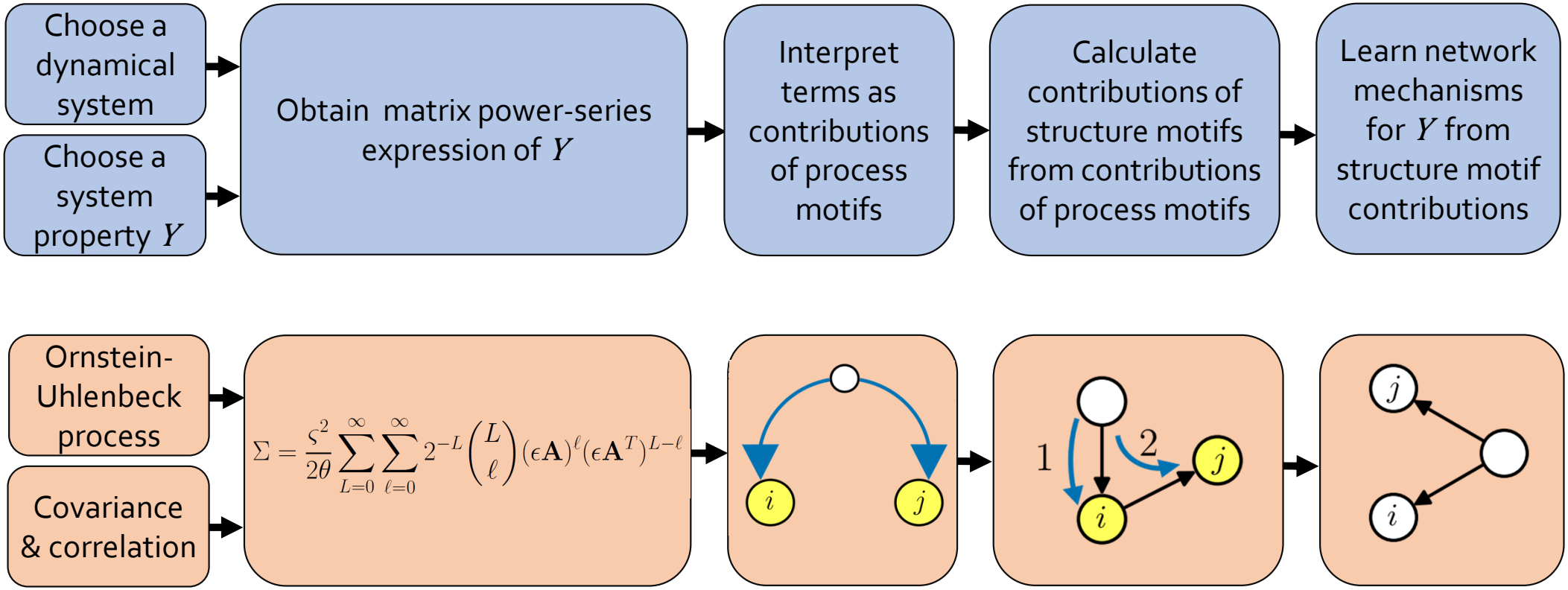
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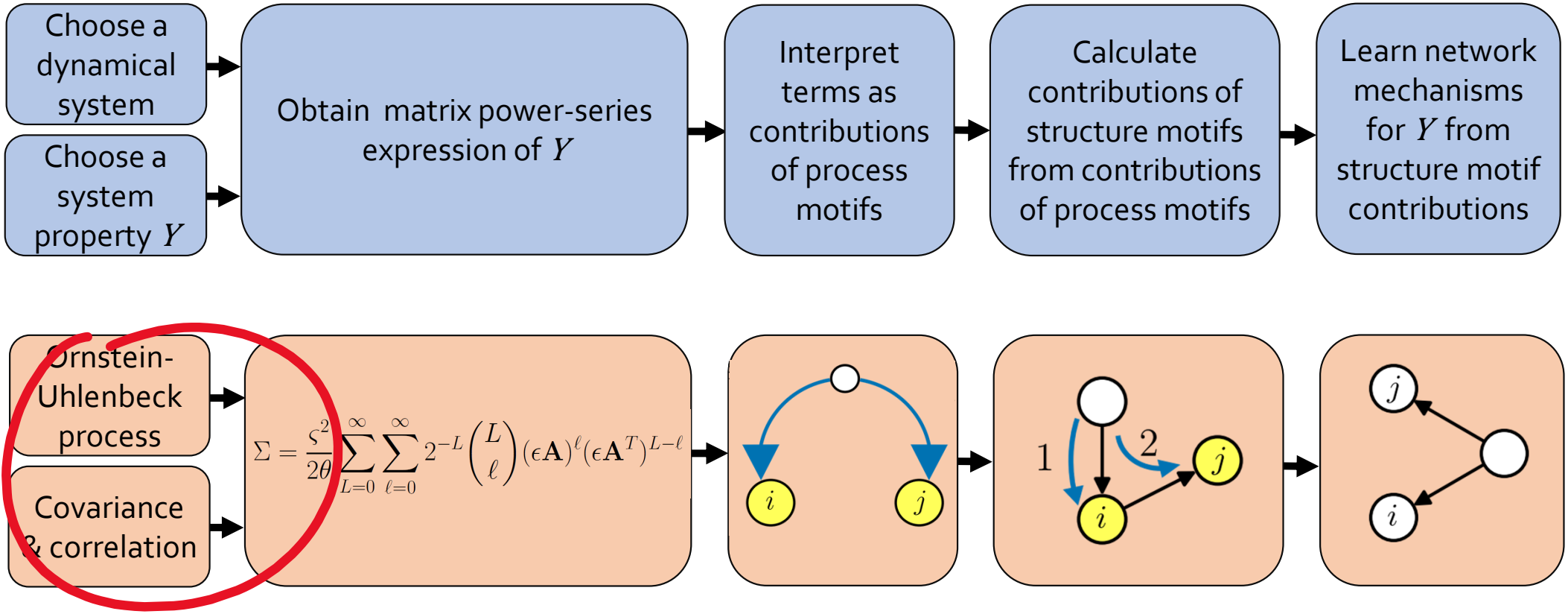
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Covariance & correlation for the OUP

Ornstein-Uhlenbeck process

Simple stochastic differential equation

Covariance and correlation

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Popular in neuroscience, econometrics, etc.

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For OU process, entropy, mutual information, etc. are functions of covariance and/or correlation

The Ornstein-Uhlenbeck process

$$d\mathbf{x}_{t+dt} = \theta(\epsilon\mathbf{A} - \mathbf{I})\mathbf{x}_t\, dt + \varsigma\, dW_t$$

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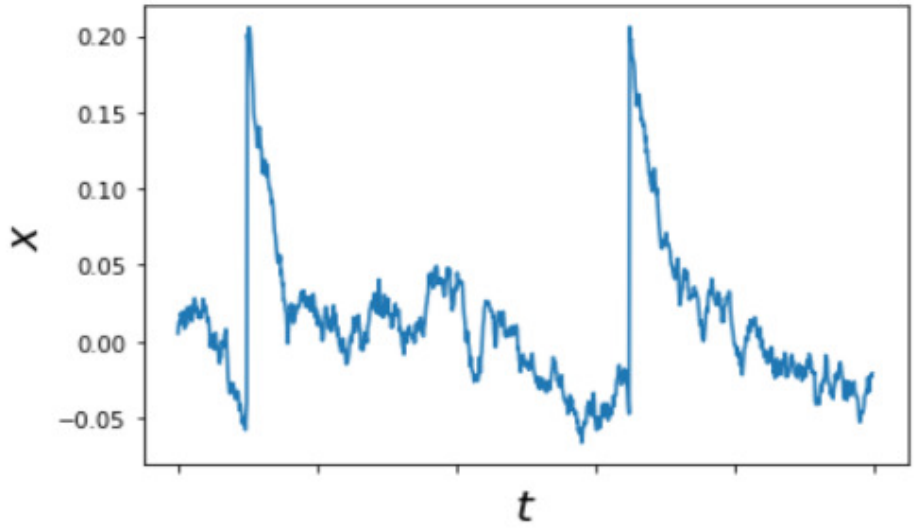

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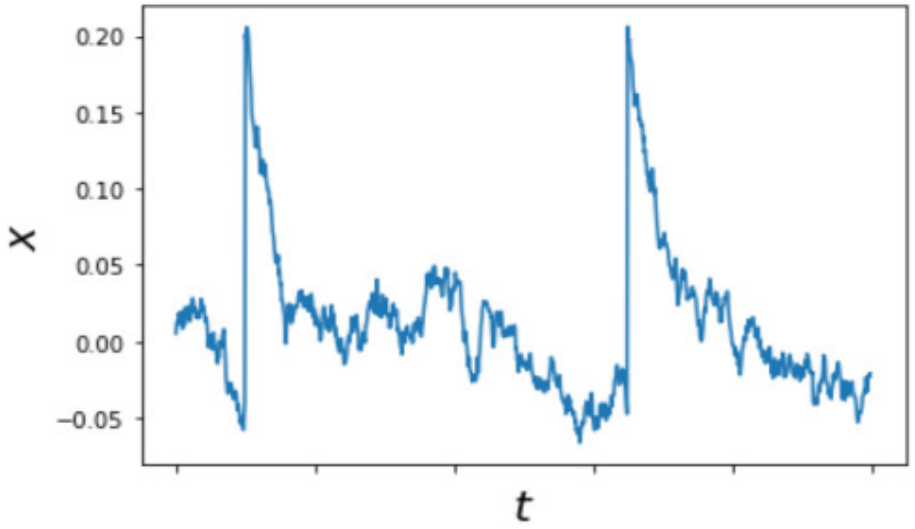

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Covariance matrix Σ

$$\begin{aligned}\boldsymbol{\Sigma} &= \langle \mathbf{x}_t\, \mathbf{x}_t^T \rangle = \langle \mathbf{x}_{t+dt}\, \mathbf{x}_{t+dt}^T \rangle \\ &= \frac{\varsigma^2}{2\theta} \sum_{L=0}^{\infty} \sum_{\ell=0}^{\infty} 2^{-L} \binom{L}{\ell} (\epsilon\mathbf{A})^{\ell} (\epsilon\mathbf{A}^T)^{L-\ell}\end{aligned}$$

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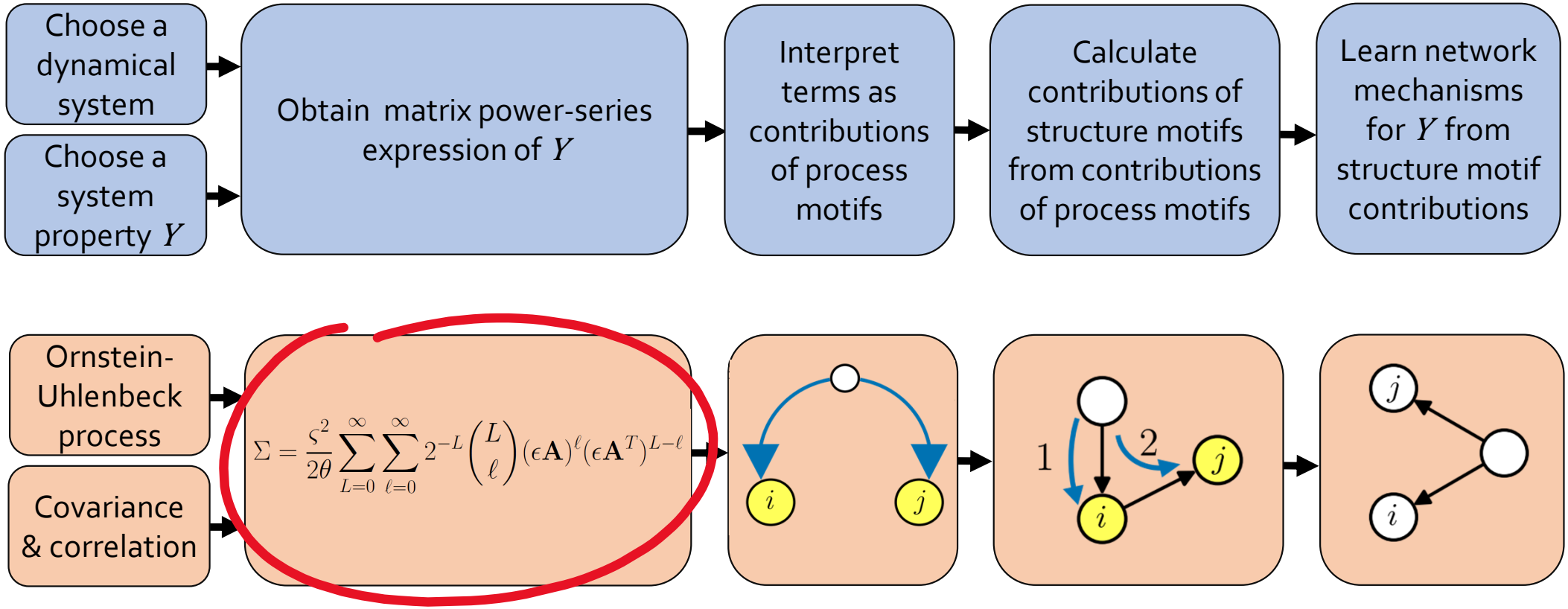
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i
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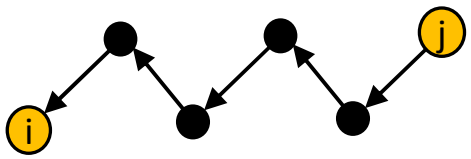
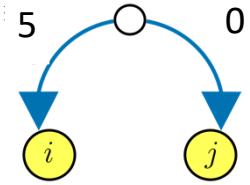
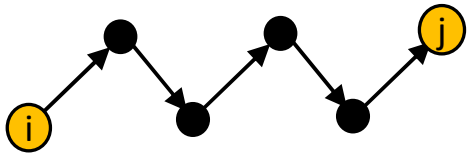
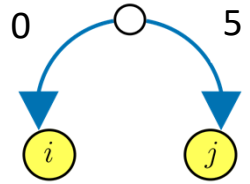
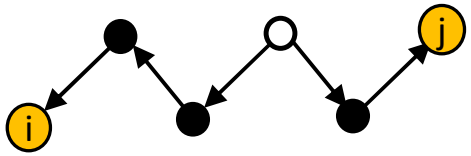
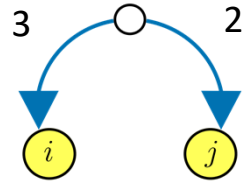
Matrix powers and walks in networks

Matrix	Edges	Walks
$\mathbf{A}^5$		
$(\mathbf{A}^T)^5$		
$\mathbf{A}^3(\mathbf{A}^T)^2$		

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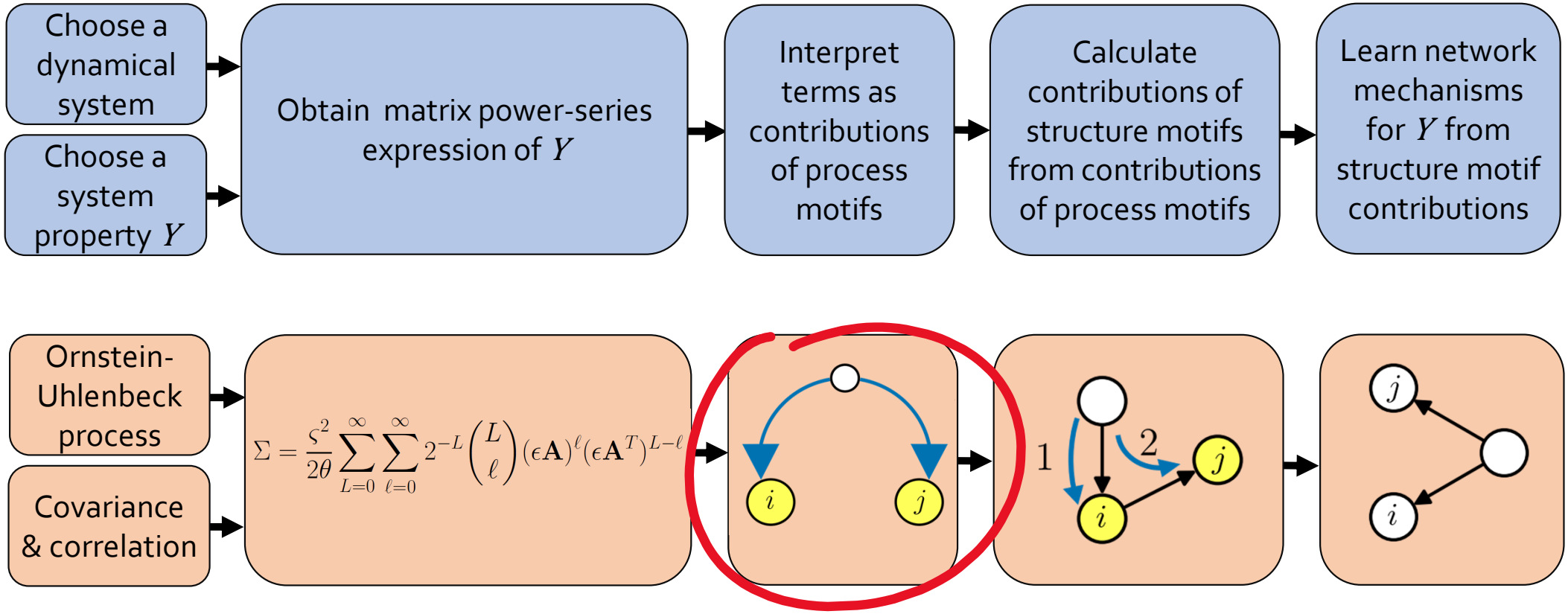
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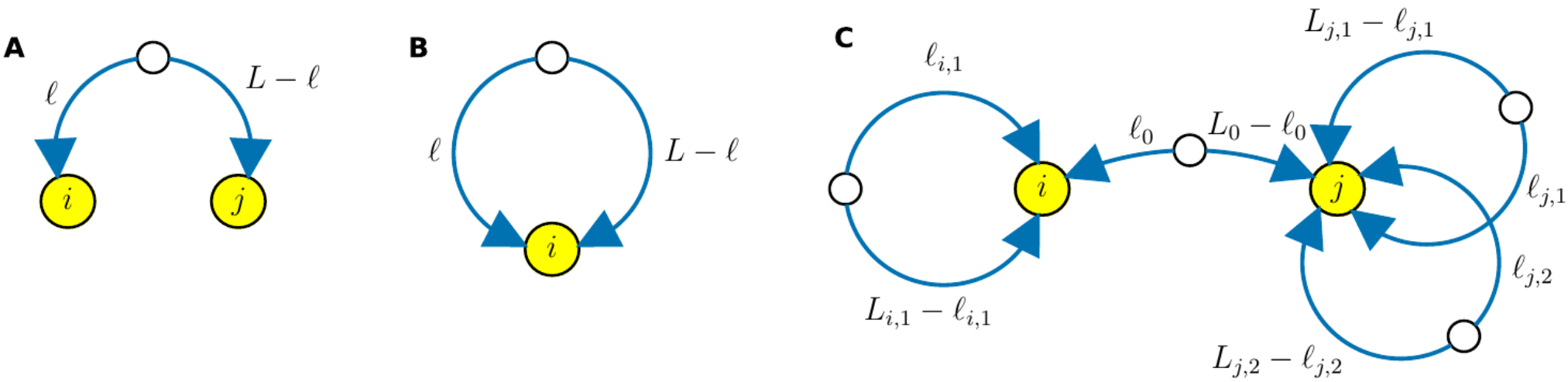
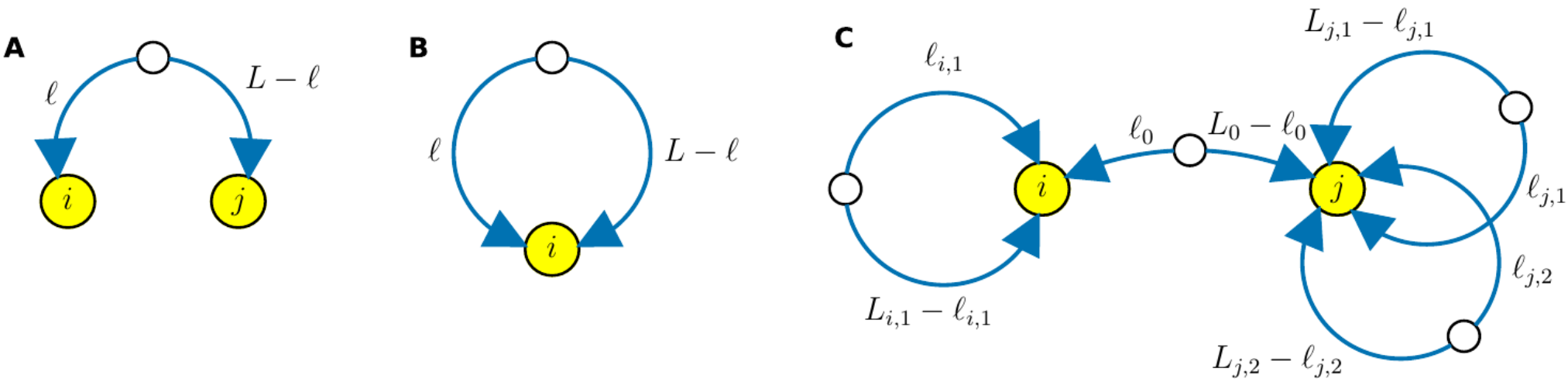
Process motifs



Process motifs for (A) covariance, (B) variance, and (C) correlation.

Process motifs



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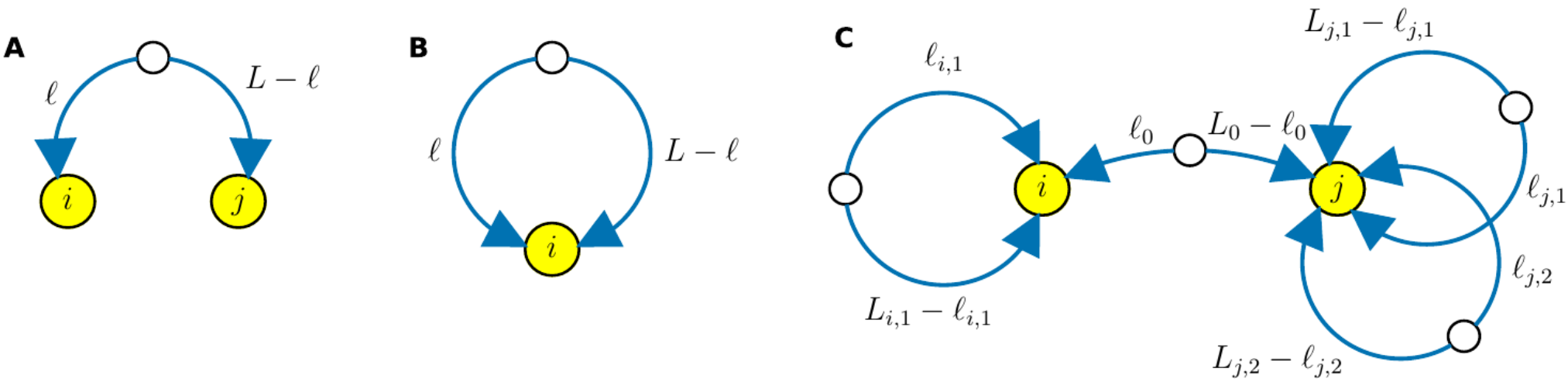

Process motifs for (A) covariance, (B) variance, and (C) correlation.

Contributions of process motifs

Process-motif contributions

$$\Sigma = \frac{\varsigma^2}{2\theta} \sum_{L=0}^{\infty} \sum_{\ell=0}^{\infty} \left(\frac{\epsilon}{2}\right)^L \binom{L}{\ell} \mathbf{A}^{\ell} (\mathbf{A}^T)^{L-\ell}$$

Structure-motif contributions

Contributions of process motifs

Process-motif contributions

count

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Structure-motif contributions

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Process-motif contributions

contribution

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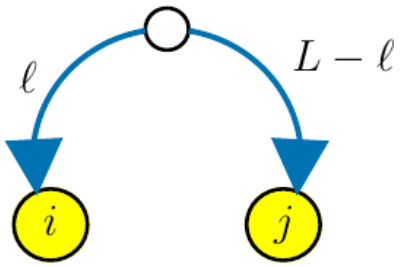
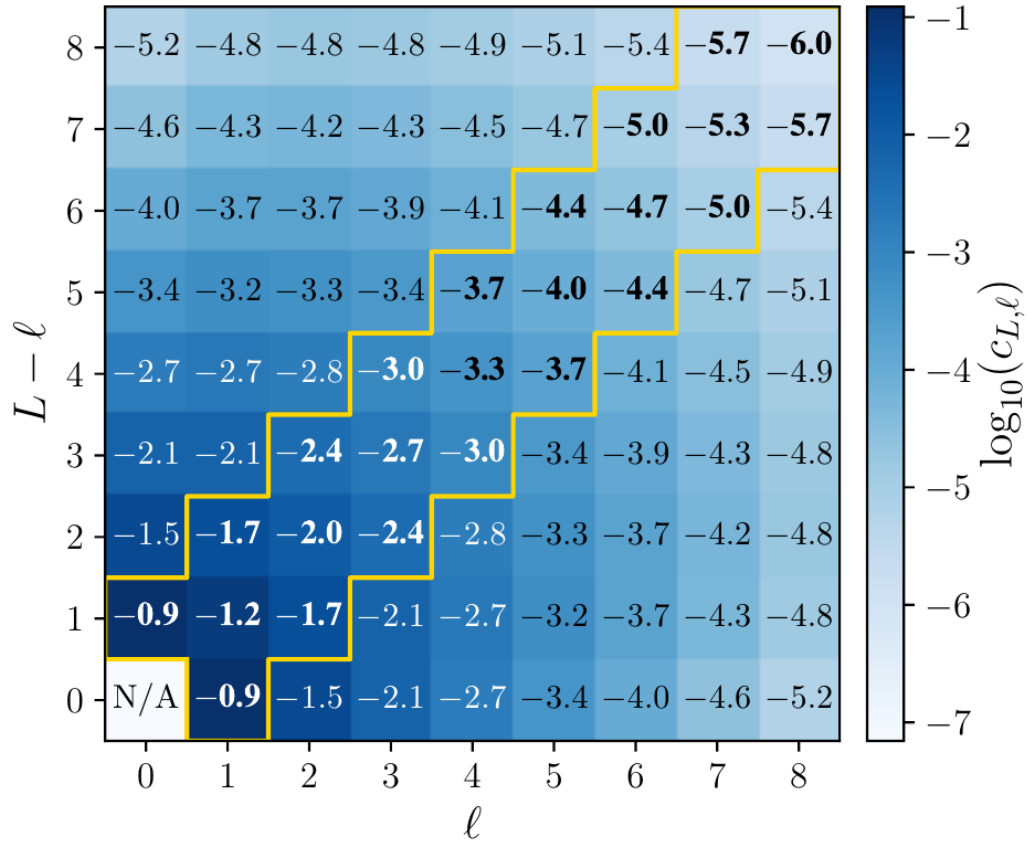
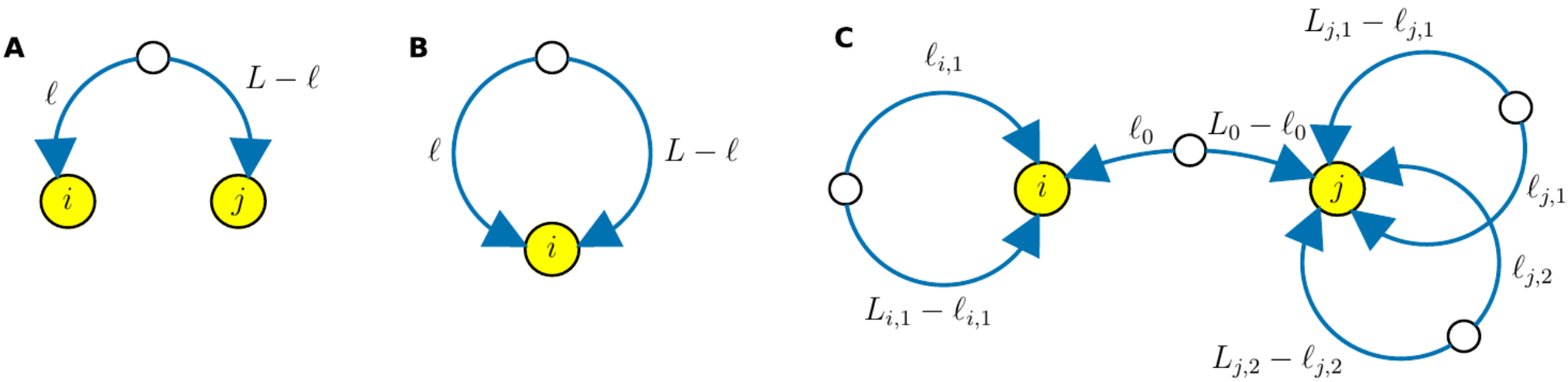
Contributions of process motifs





Process motifs (recap)



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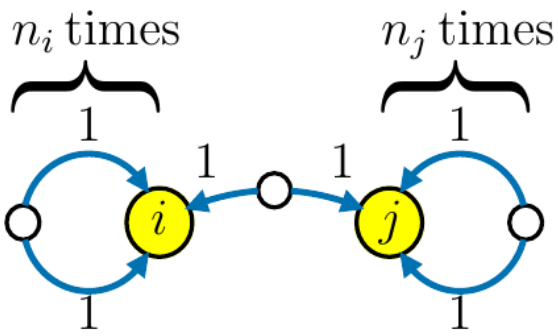
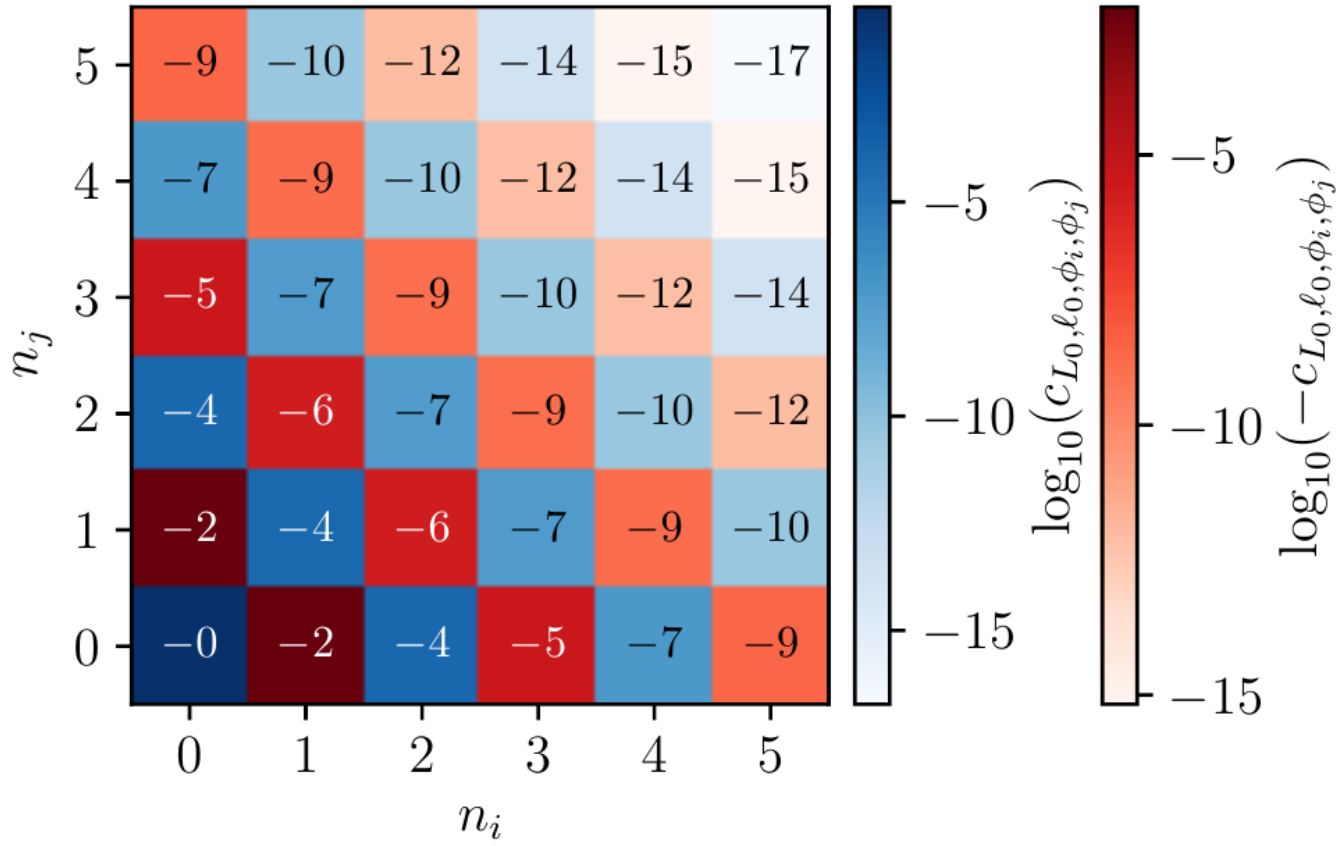
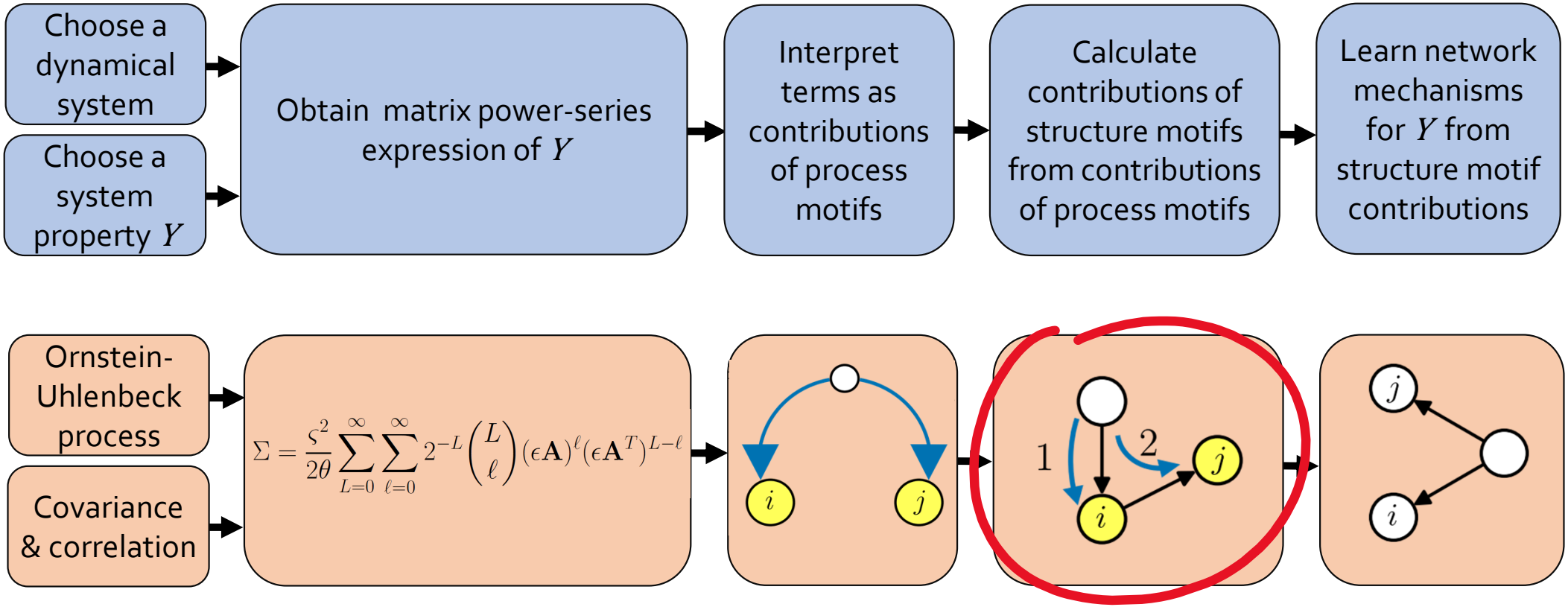




Pipeline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Calculate contributions of structure motifs from contributions of process motifs
Learn network mechanisms for Y from structure motif contributions
Ornstein-Uhlenbeck process
Covariance & correlation
$\Sigma = \frac{\varsigma^2}{2\theta} \sum_{L=0}^{\infty} \sum_{\ell=0}^{\infty} 2^{-L} \binom{L}{\ell} (\epsilon \mathbf{A})^{\ell} (\epsilon \mathbf{A}^T)^{L-\ell}$
i
j
1
2
j
i
j
i

Specific contributions of structure motifs

Process-motif contributions

contribution

count

$$\Sigma = \frac{\zeta^2}{2\theta} \sum_{L=0}^{\infty} \sum_{\ell=0}^{\infty} \left(\frac{\epsilon}{2}\right)^L \binom{L}{\ell} \mathbf{A}^{\ell} (\mathbf{A}^T)^{L-\ell}$$

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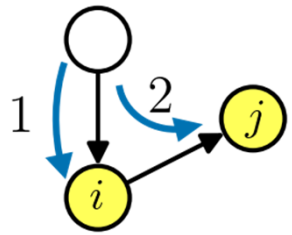
Structure-motif contributions

total contribution

$$c_s = \sum_{p_i \text{ “on“ } s} b_{p_i}$$



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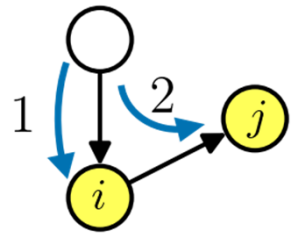

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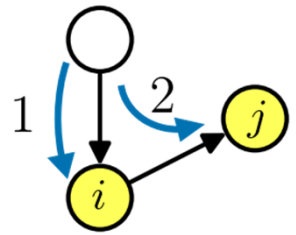

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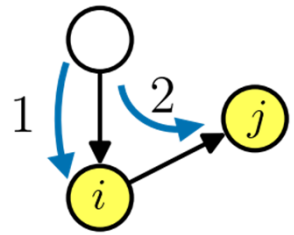

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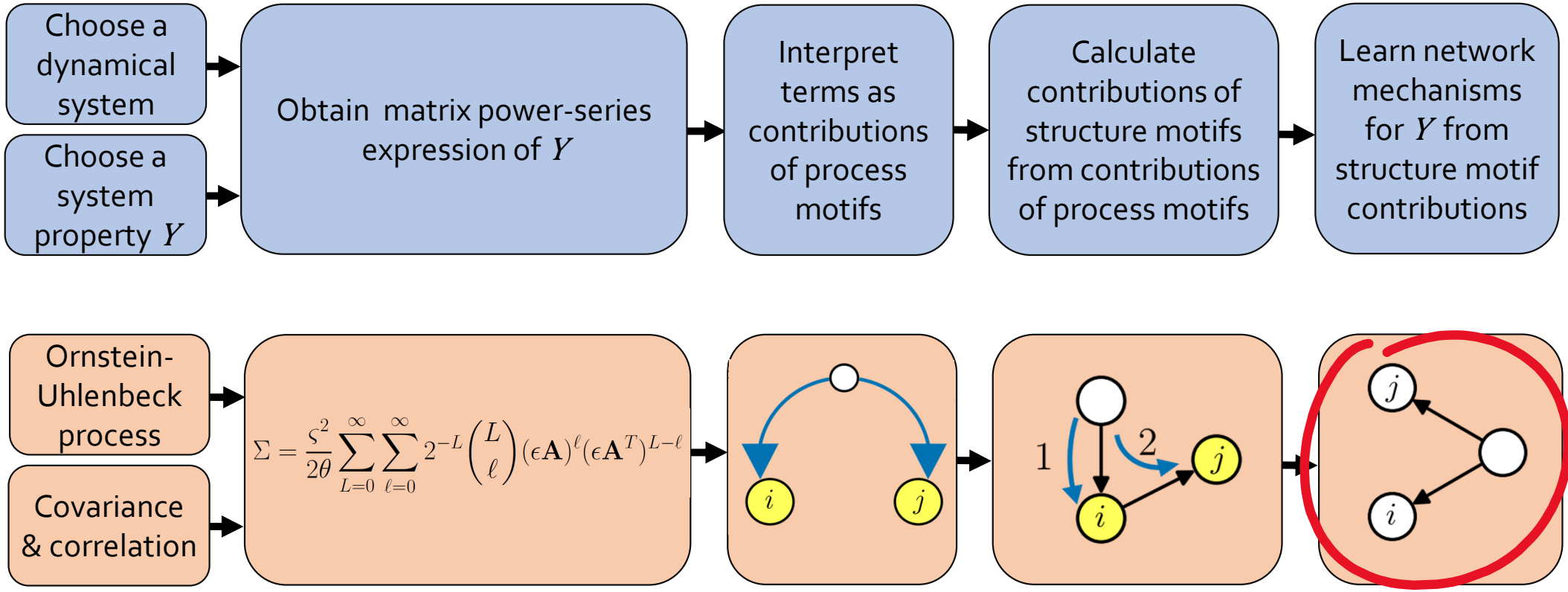
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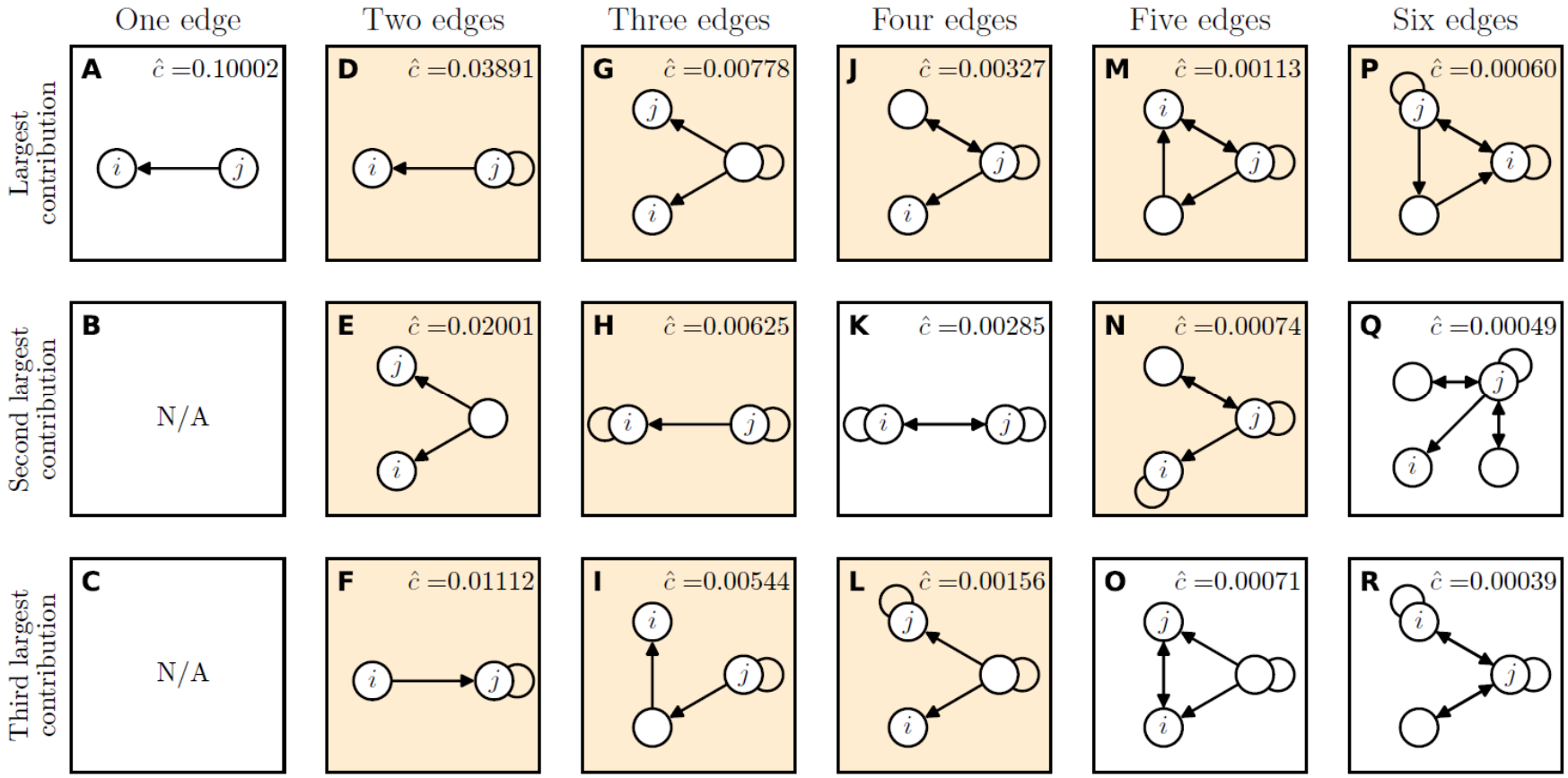
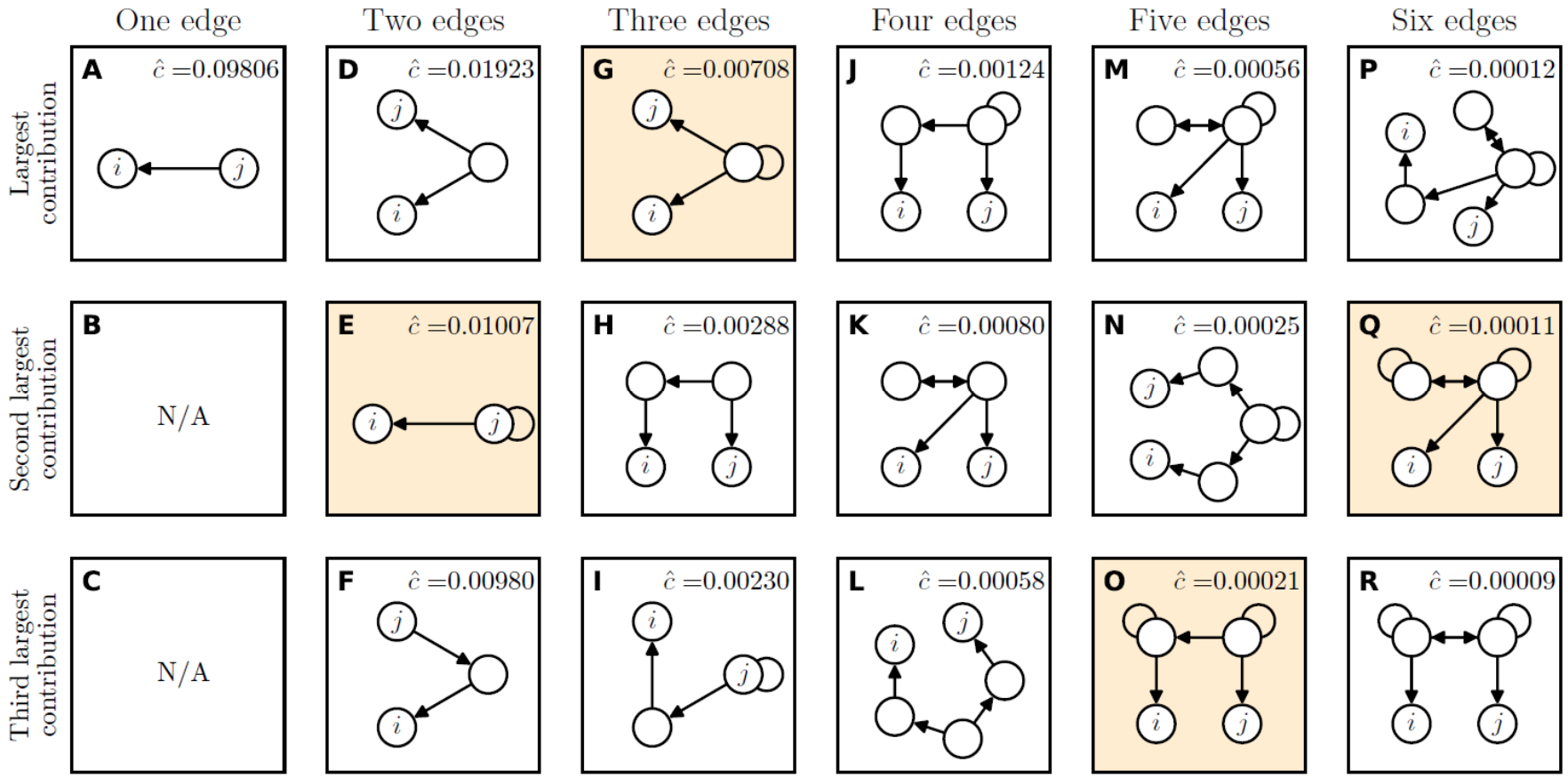
Contributions to covariance

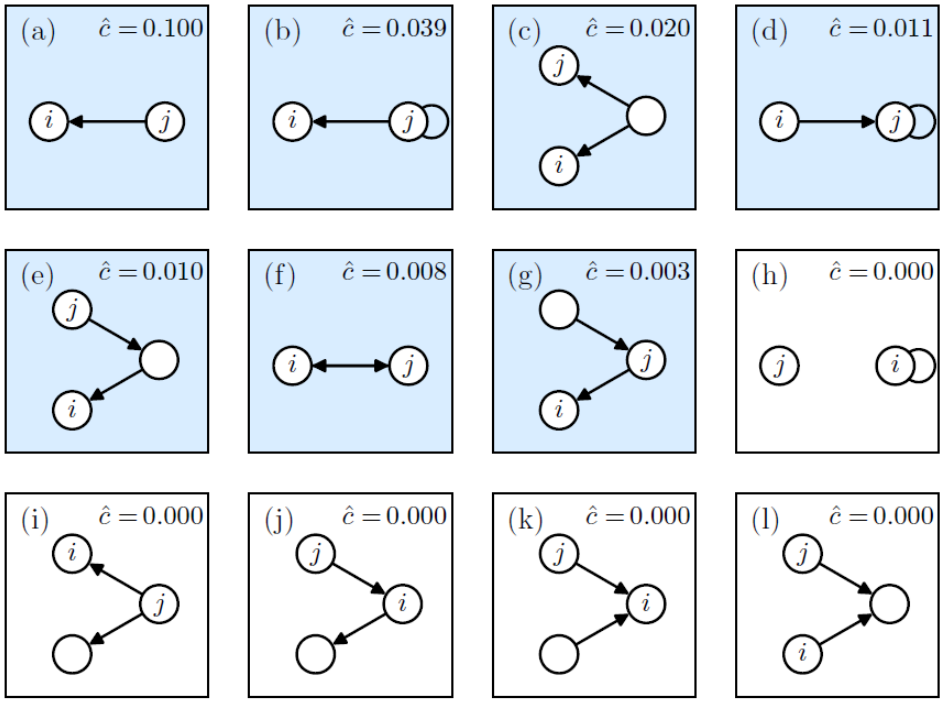


Contributions to correlation

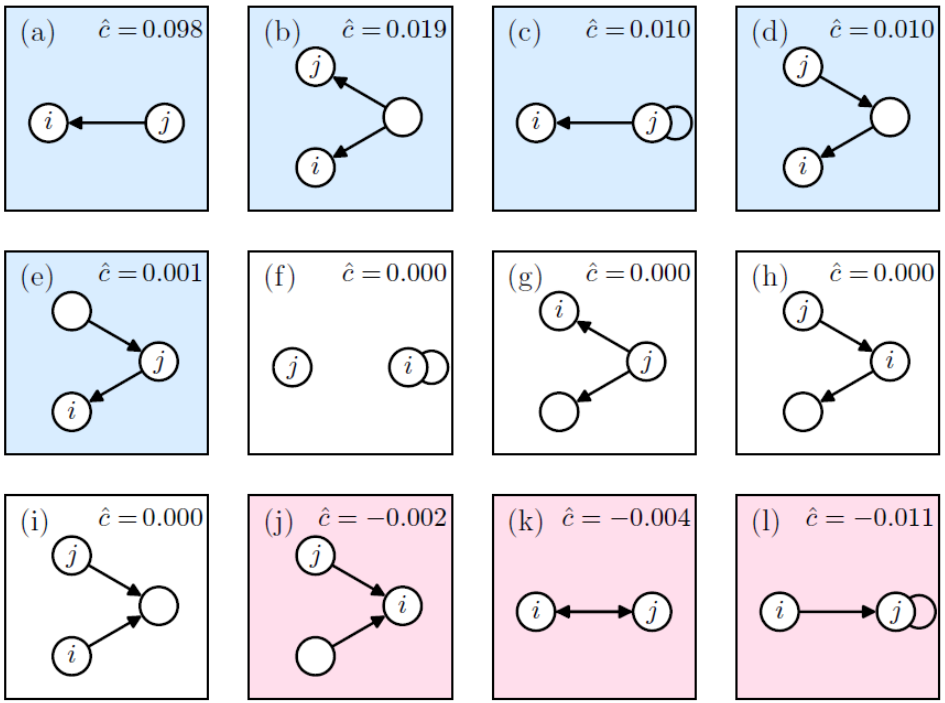


Emergence in 2-edge motifs

Covariance



Correlation



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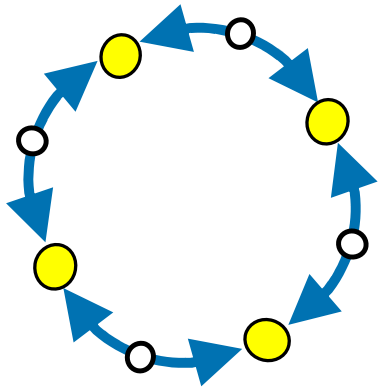
- Tool to explore the **importance of recurrence** for dynamics on networks for neuroscience, ML, epidemiology, opinion dynamics, etc.

Plugs!

- More talks!
  - Process motifs for entropy

    → Tuesday September 22, 18:15 (Session 7B)



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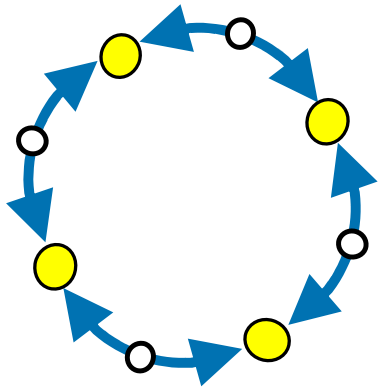

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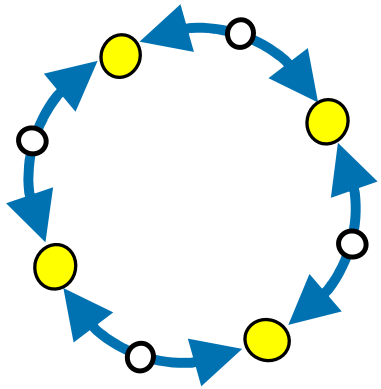

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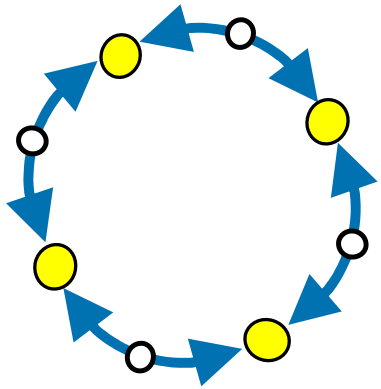

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