Motifs for processes on networks

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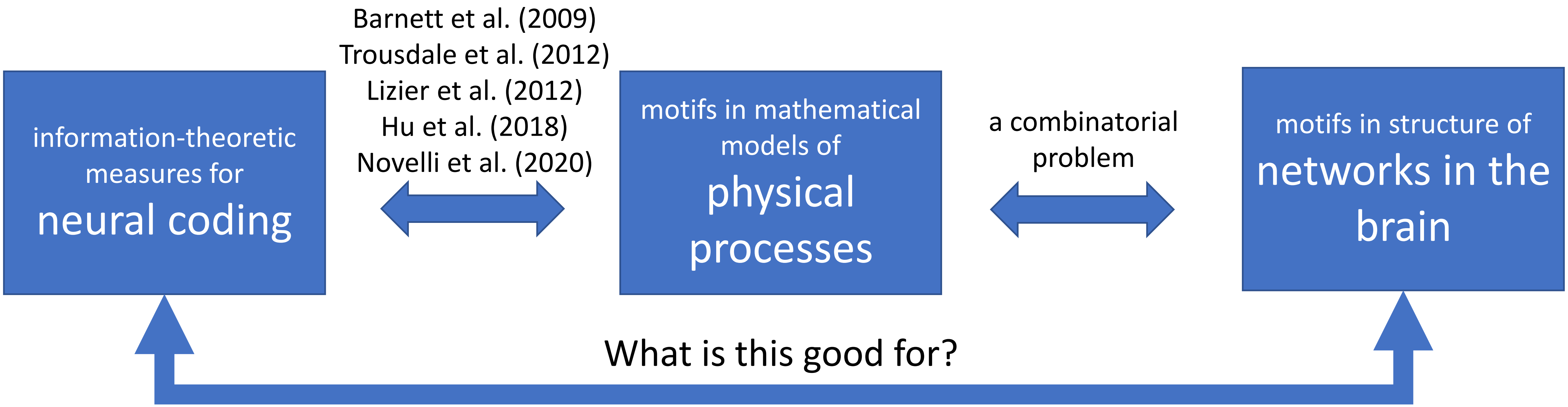


Connection between information-theoretic measures and network structure
Physical interpretation of information-theoretic measures of neural activity
Theoretical underpinning for relationship between structural and functional connectivity

1-slide summary

Outline

1. Pipeline
2. An example: Covariance and correlation for the OU process
    1. Why care about this example?
    2. OU process
    3. Process motifs
    4. Structure motifs
3. Conclusions

Choose a dynamical system

Choose a system property Y

Obtain matrix power-series expression of Y

Interpret terms as contributions of process motifs

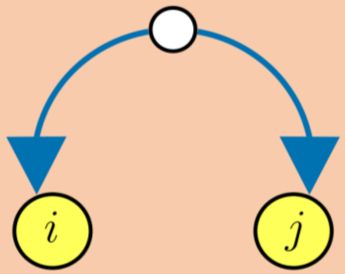
Calculate contributions of structure motifs from contributions of process motifs

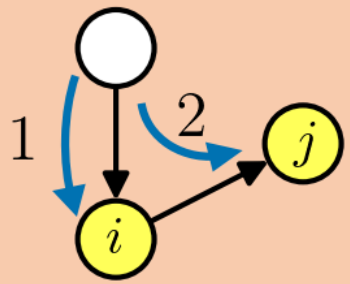
Learn network mechanisms for Y from structure motif contributions

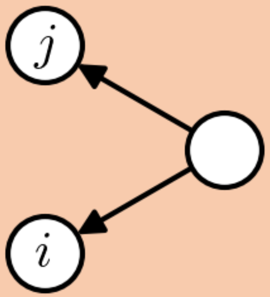
Ornstein-Uhlenbeck process

Covariance & correlation

$$\Sigma = \frac{\varsigma^2}{2\theta}\sum_{L=0}^{\infty}\sum_{\ell=0}^{\infty} 2^{-L}\binom{L}{\ell}(\epsilon\mathbf{A})^{\ell}(\epsilon\mathbf{A}^T)^{L-\ell}$$







Pipeline

Pipeline

Choose a dynamical system → Obtain matrix power-series expression of Y

Choose a system property Y → Obtain matrix power-series expression of Y

Obtain matrix power-series expression of Y → Interpret terms as contributions of process motifs → Calculate contributions of structure motifs from contributions of process motifs → Learn network mechanisms for Y from structure motif contributions

Ornstein-Uhlenbeck process

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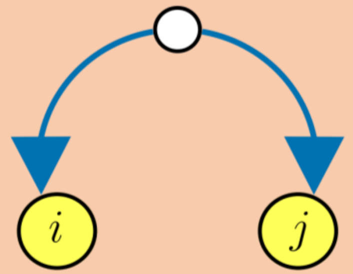
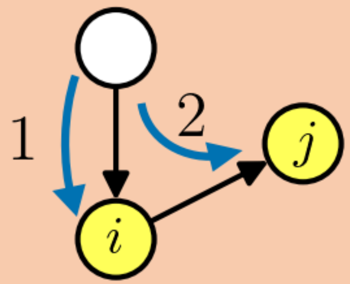
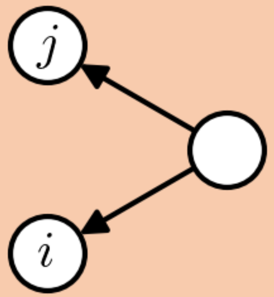






Ornstein-Uhlenbeck process

Simple stochastic differential equation

Popular in neuroscience, econometrics, etc.

Linear-response approximation of IF model

Covariance and correlation

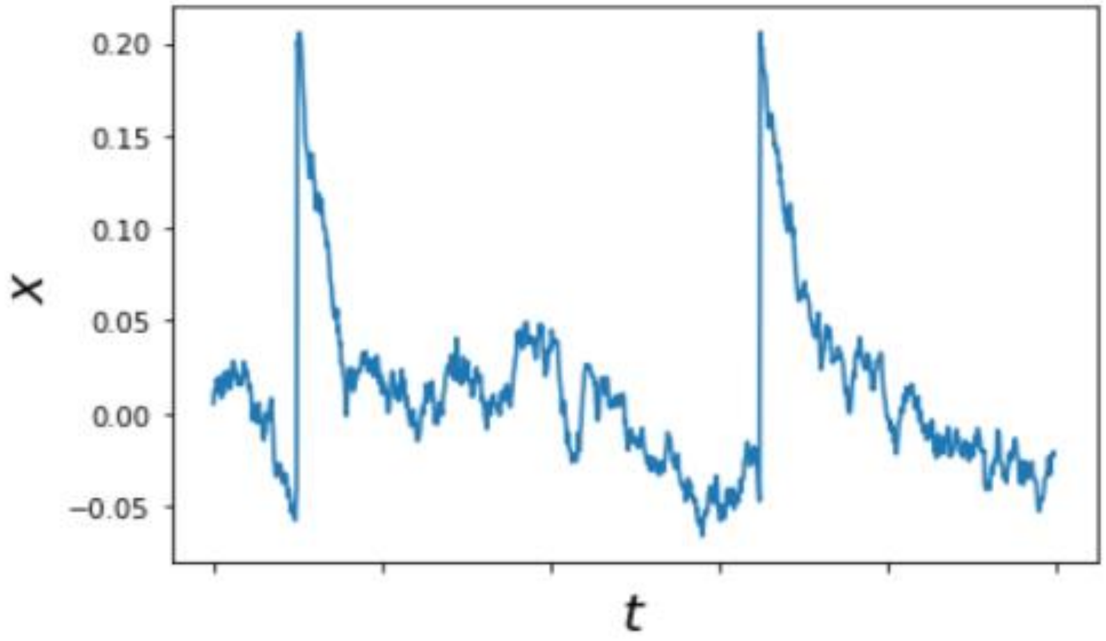
Simple measure of interaction for pairs of variables

Popular measure of functional connectivity

For OU process, entropy, mutual information, etc. are functions of covariance and/or correlation

Covariance & correlation for OUP

$$d\mathbf{x}_{t+dt} = \theta(\epsilon\mathbf{A} - \mathbf{I})\mathbf{x}_t \, dt + \varsigma \, dW_t$$



The Ornstein-Uhlenbeck process

$$d\mathbf{x}_{t+dt} = \theta(\epsilon\mathbf{A} - \mathbf{I})\mathbf{x}_t\,dt + \varsigma\,dW_t$$

Covariance matrix Σ

$$\begin{aligned}\boldsymbol{\Sigma} &= \langle \mathbf{x}_t\,\mathbf{x}_t^T \rangle = \langle \mathbf{x}_{t+dt}\,\mathbf{x}_{t+dt}^T \rangle \\ &= \frac{\varsigma^2}{2\theta} \sum_{L=0}^{\infty} \sum_{\ell=0}^{\infty} 2^{-L} \binom{L}{\ell} (\epsilon\mathbf{A})^{\ell} (\epsilon\mathbf{A}^T)^{L-\ell}\end{aligned}$$

The Ornstein-Uhlenbeck process

Pipeline (recap)

Choose a dynamical system

Choose a system property Y

→ Obtain matrix power-series expression of Y → Interpret terms as contributions of process motifs → Calculate contributions of structure motifs from contributions of process motifs → Learn network mechanisms for Y from structure motif contributions

Ornstein-Uhlenbeck process

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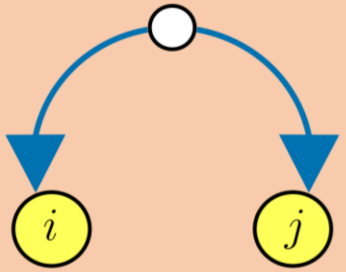
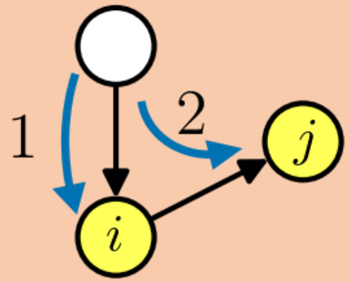
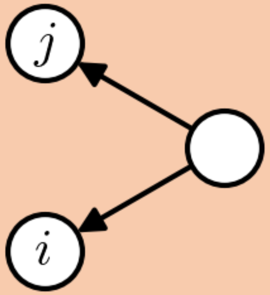
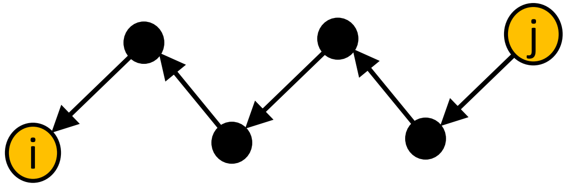
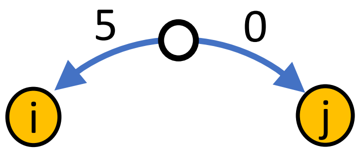
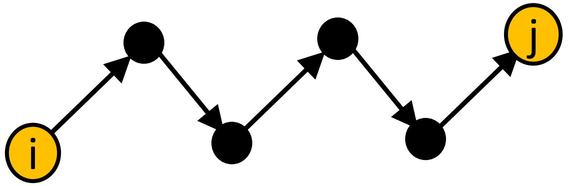
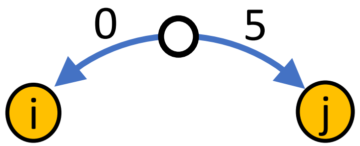
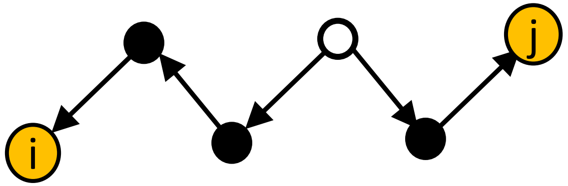
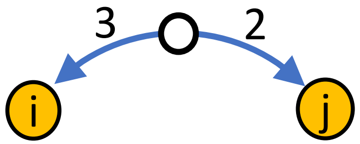






Matrix	Edge diagramme	Walk diagramme
$\mathbf{A}^5$		
$(\mathbf{A}^{\mathrm{T}})^5$		
$\mathbf{A}^3(\mathbf{A}^{\mathrm{T}})^2$		

Matrix powers and walks in networks

Pipeline (recap #2)

Choose a dynamical system

Choose a system property Y

Obtain matrix power-series expression of Y

Interpret terms as contributions of process motifs

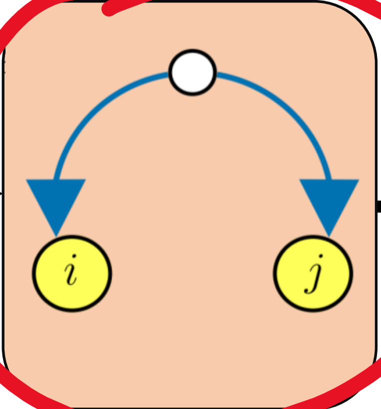
Calculate contributions of structure motifs from contributions of process motifs

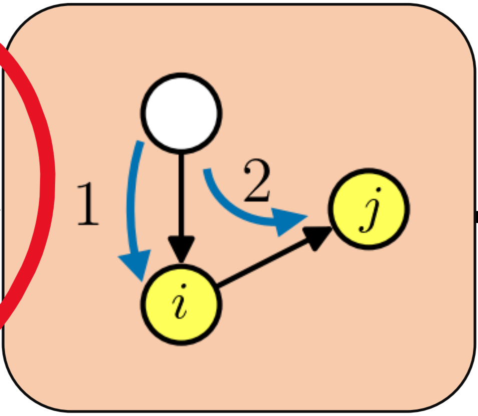
Learn network mechanisms for Y from structure motif contributions

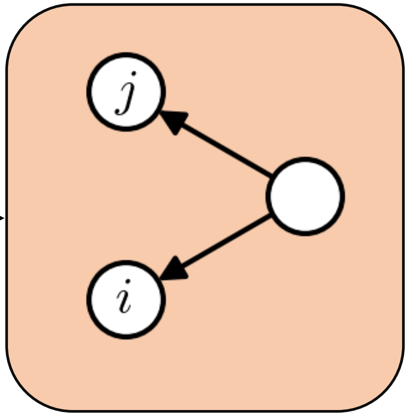
Ornstein-Uhlenbeck process

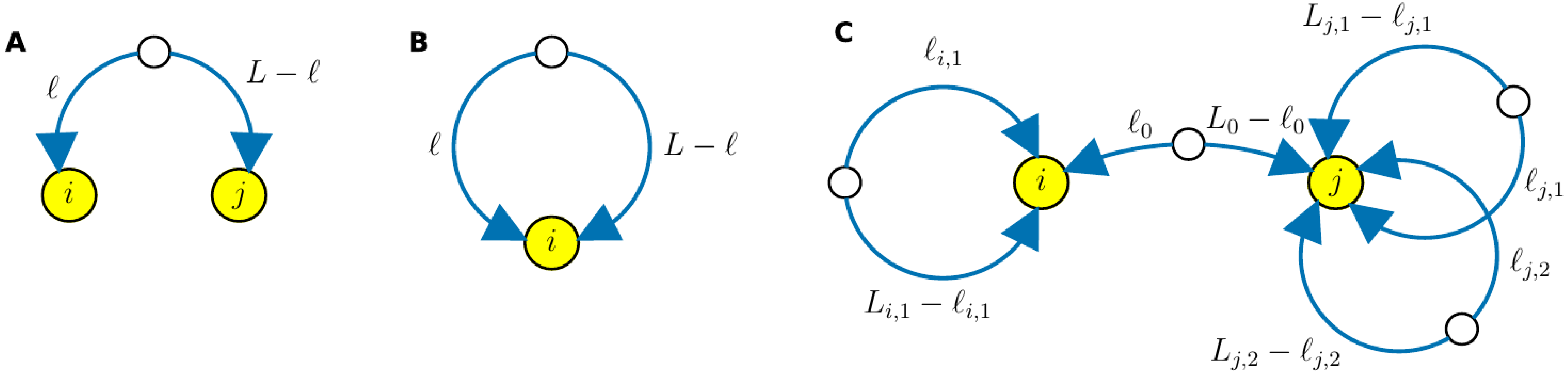
Covariance & correlation

$$\Sigma = \frac{\varsigma^2}{2\theta}\sum_{L=0}^{\infty}\sum_{\ell=0}^{\infty} 2^{-L}\binom{L}{\ell}(\epsilon \mathbf{A})^{\ell}(\epsilon \mathbf{A}^T)^{L-\ell}$$









Process motifs for (A) covariance, (B) variance, and (C) correlation.

Process-motif contributions

contribution

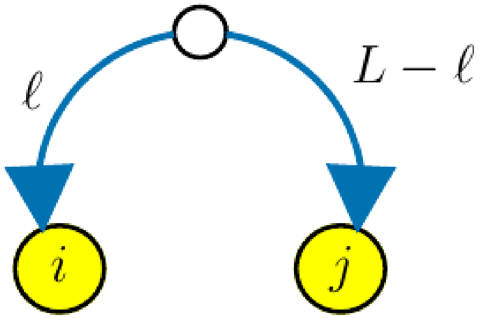
count

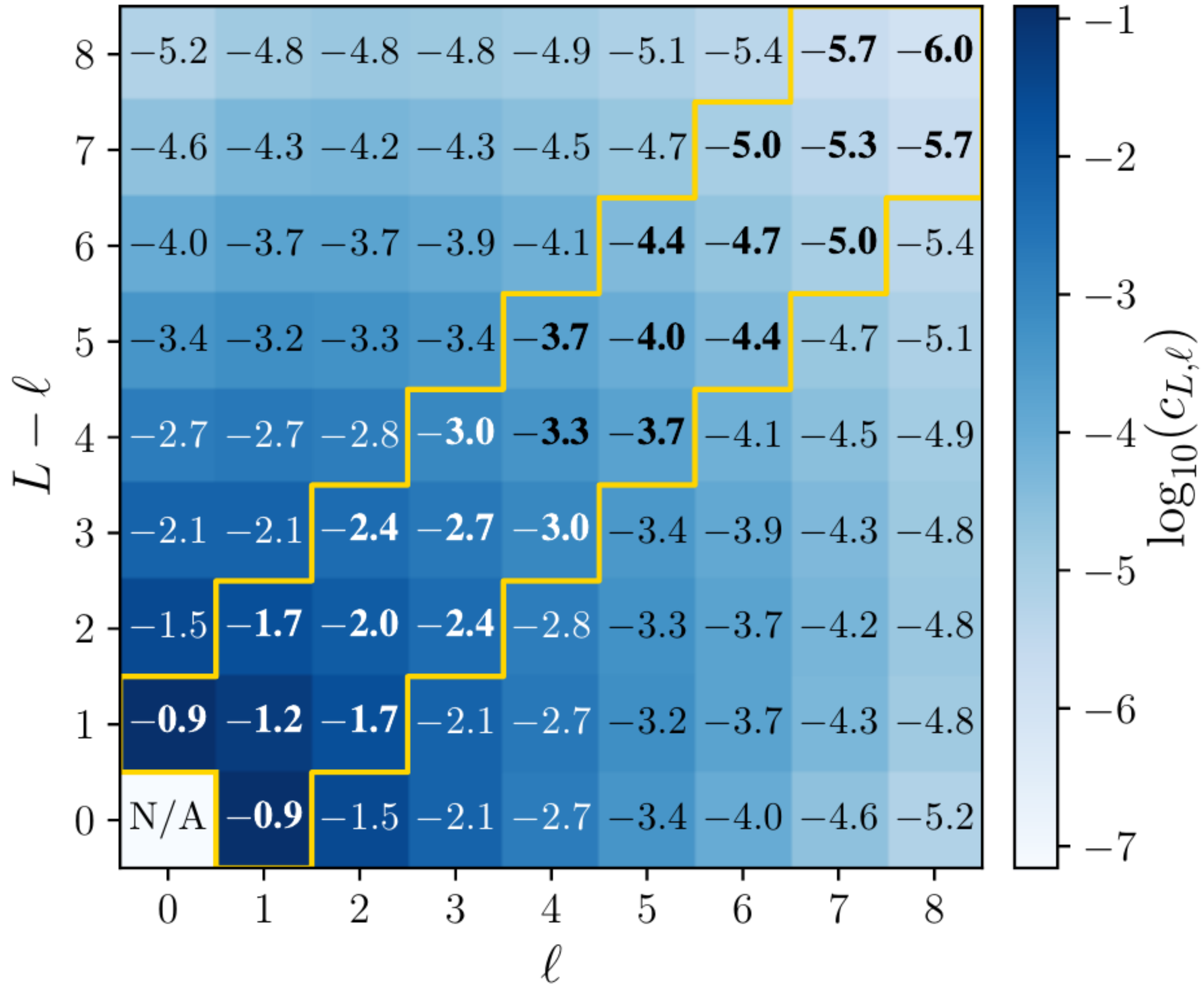
$$\Sigma = \frac{\varsigma^2}{2\theta} \sum_{L=0}^{\infty} \sum_{\ell=0}^{\infty} \left(\frac{\epsilon}{2}\right)^L \binom{L}{\ell} \mathbf{A}^\ell (\mathbf{A}^T)^{L-\ell}$$

$$\Sigma = \sum_{i=1}^{k} b_{p_i} n_{p_i}$$

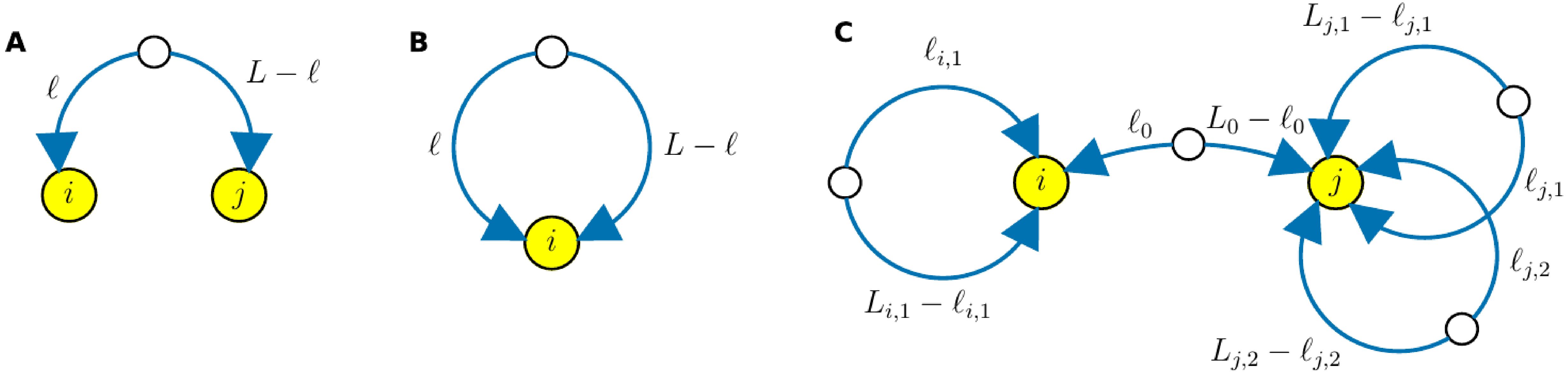
Structure-motif contributions

Contributions of process motifs



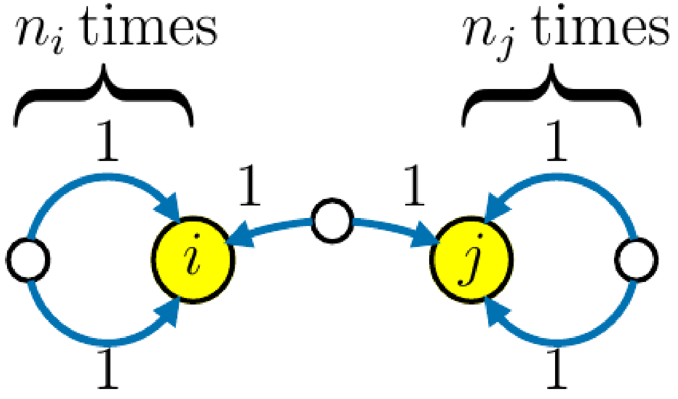


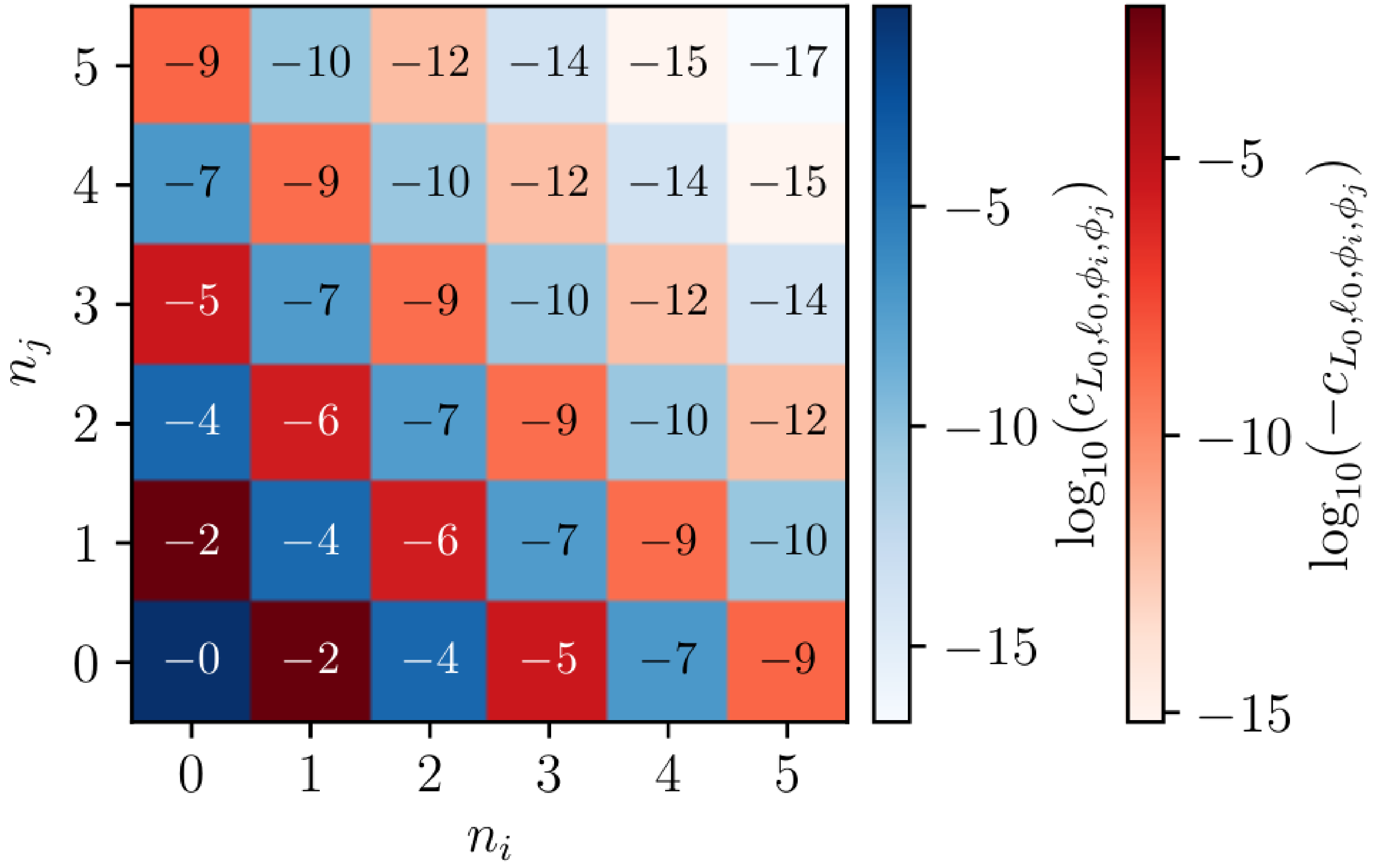
Contributions of process motifs



Process motifs for (A) covariance, (B) variance, and (C) correlation.

Process motifs (recap)





Contributions of process motifs

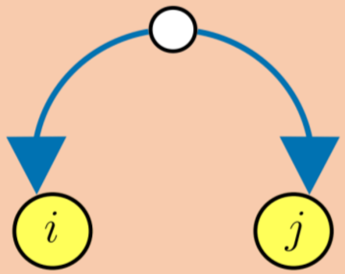
Pipeline (recap #3)

Choose a dynamical system → Choose a system property Y → Obtain matrix power-series expression of Y → Interpret terms as contributions of process motifs → Calculate contributions of structure motifs from contributions of process motifs → Learn network mechanisms for Y from structure motif contributions

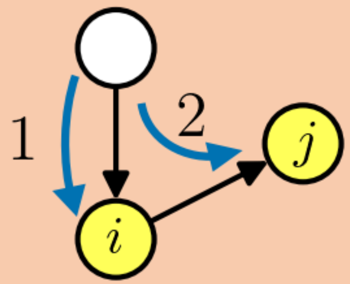
Ornstein-Uhlenbeck process → Covariance & correlation →

$$\Sigma = \frac{\varsigma^2}{2\theta}\sum_{L=0}^{\infty}\sum_{\ell=0}^{\infty}2^{-L}\binom{L}{\ell}(\epsilon\mathbf{A})^{\ell}(\epsilon\mathbf{A}^T)^{L-\ell}$$

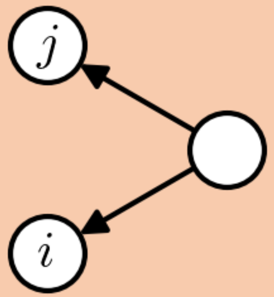
→



→



→



Process-motif contributions

contribution

count

$$\Sigma = \frac{\varsigma^2}{2\theta} \sum_{L=0}^{\infty} \sum_{\ell=0}^{\infty} \left(\frac{\epsilon}{2}\right)^L \binom{L}{\ell} \mathbf{A}^{\ell} (\mathbf{A}^T)^{L-\ell}$$

$$\Sigma = \sum_{i=1}^{k} b_{p_i} n_{p_i}$$

Structure-motif contributions

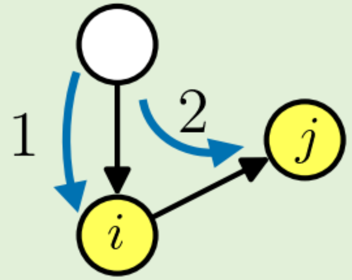
total contribution

$$c_s = \sum_{p_i \text{ ``on'' } s} b_{p_i}$$

specific contribution

$$\hat{c}_s = c_s - \sum_{t \subset s} \hat{c}_t$$

$$\Sigma = \sum_i \hat{c}_s n_{s_i}$$



Specific contributions of structure motifs

Pipeline (recap #4)

Choose a dynamical system

Choose a system property Y

Obtain matrix power-series expression of Y

Interpret terms as contributions of process motifs

Calculate contributions of structure motifs from contributions of process motifs

Learn network mechanisms for Y from structure motif contributions

Ornstein-Uhlenbeck process

Covariance & correlation

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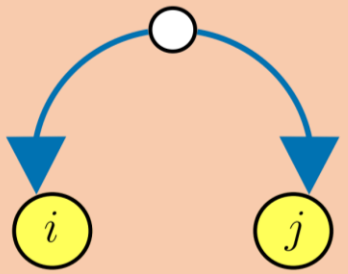
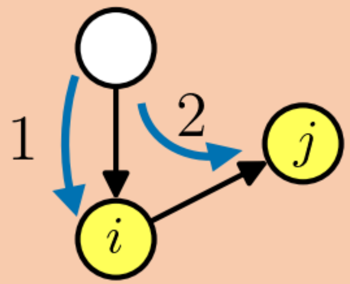
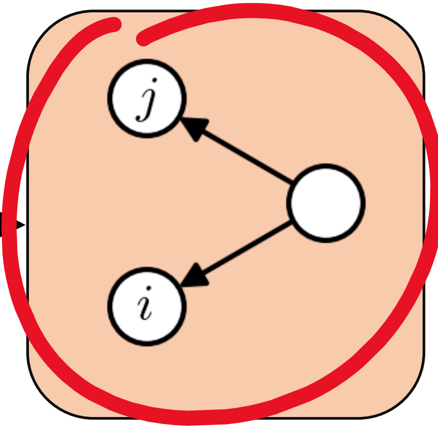
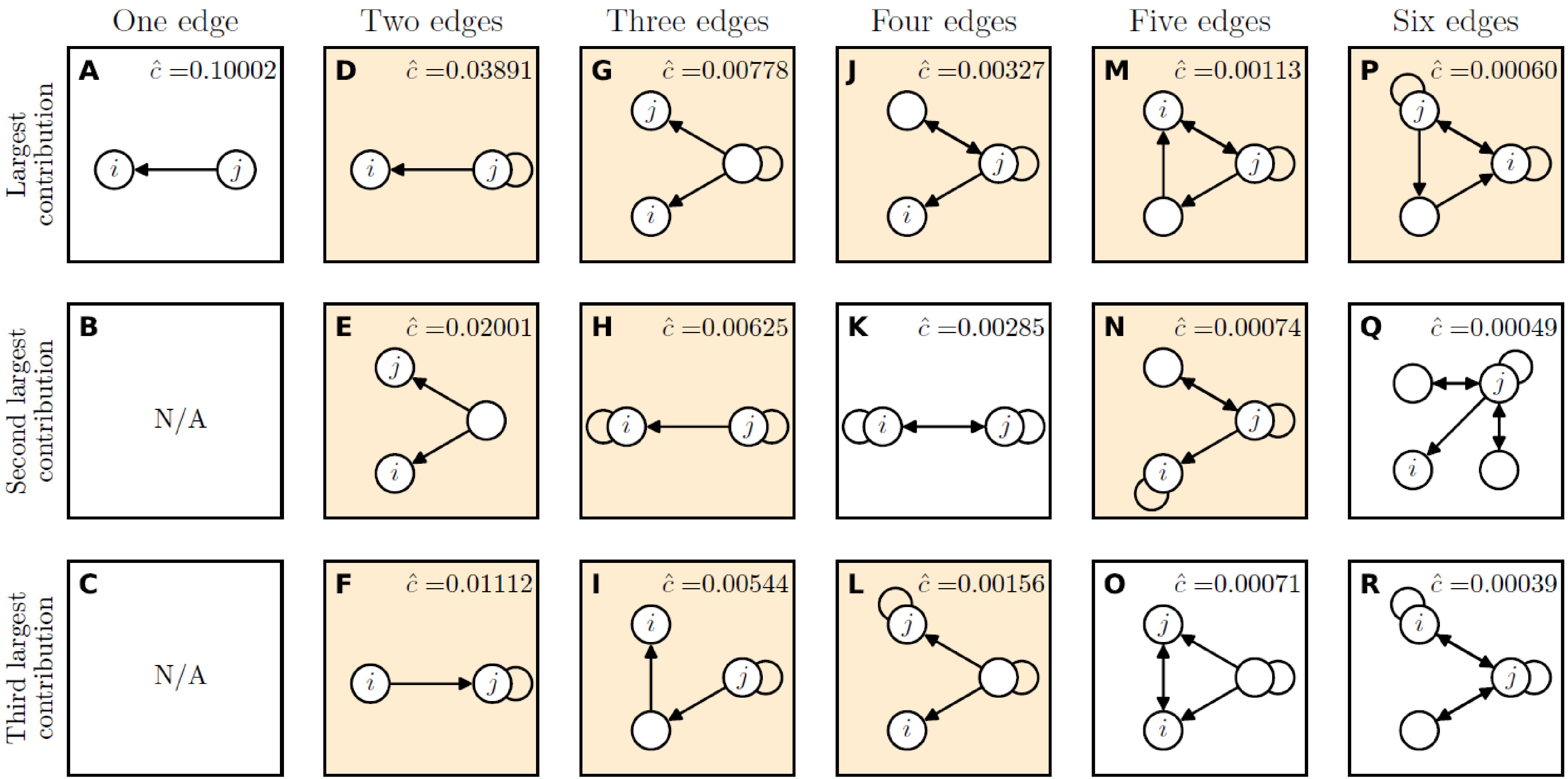
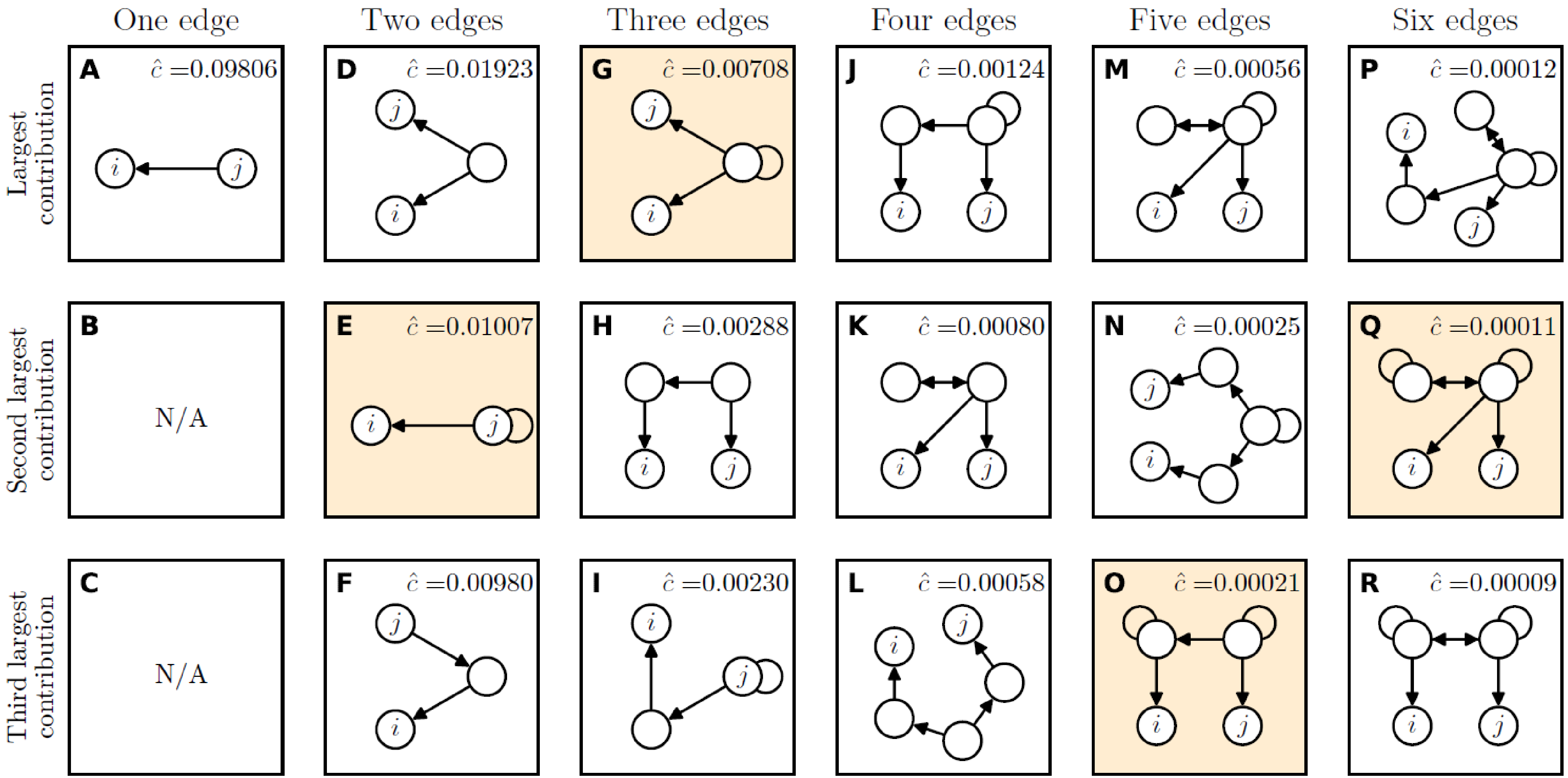








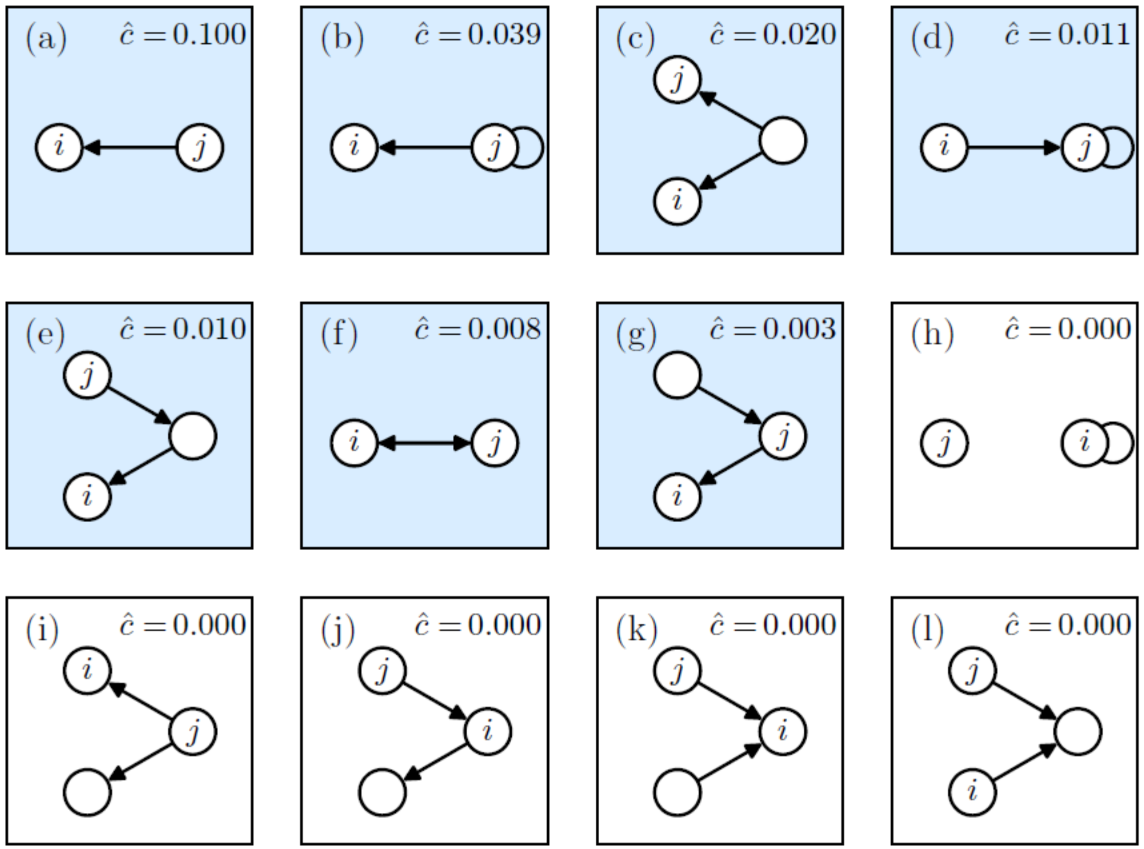
Contributions of structure motifs to covariance



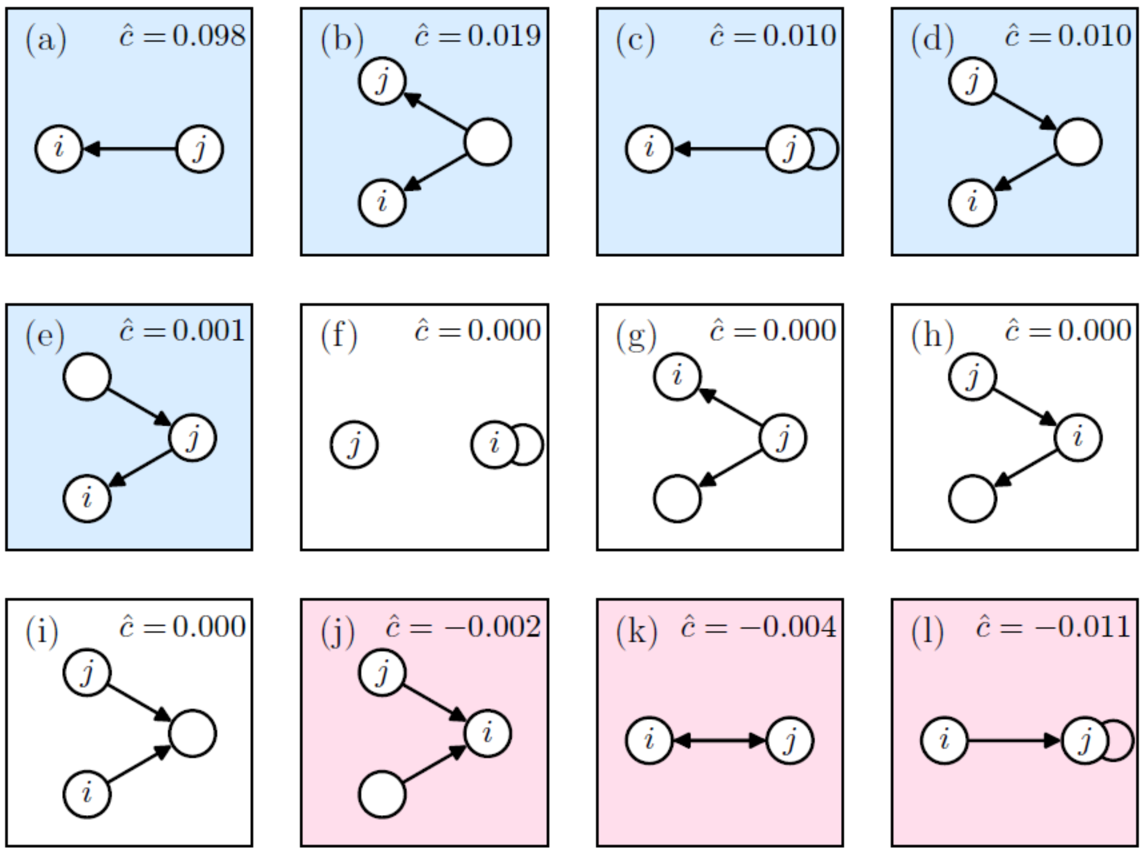
Contributions of structure motifs to correlation

2-edge “network mechanisms”

Covariance



Correlation



information-theoretic measures for neural coding

Barnett et al. (2009)
Trousdale et al. (2012)
Lizier et al. (2012)
Hu et al. (2018)
Novelli et al. (2020)

motifs in mathematical models of physical processes

a combinatorial problem

motifs in structure of networks in the brain

What is this good for?

Connection between information-theoretic measures and network structure
Physical interpretation of information-theoretic measures of neural activity
Theoretical underpinning for relationship between structural and functional connectivity

Conclusions

- Physical interpretation of information-theoretic measures via dynamical systems + graph theory approach

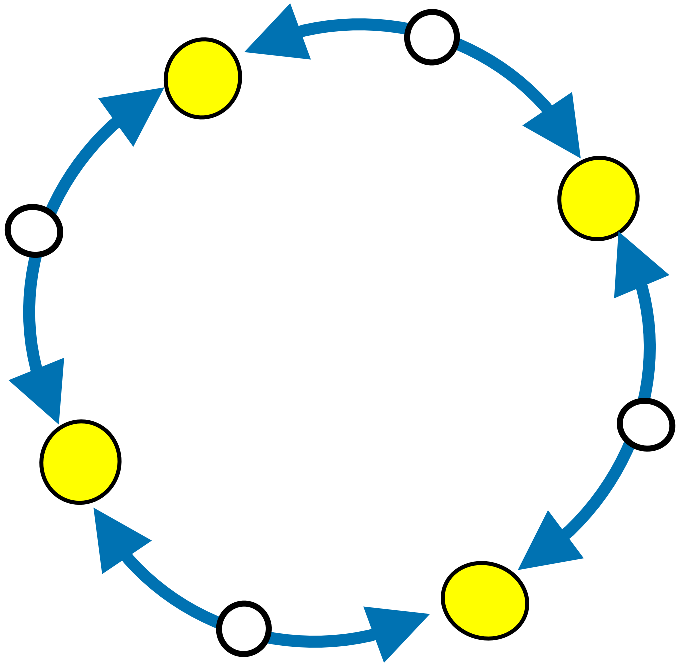
- Clear, quantitative link between network structure and information-theoretic measures

- Network mechanisms that enhance or reduce information-theoretic measures

Conclusions

Future directions

- Entropy and mutual information
- Other information-theoretic measures
- Graphical models for network inference



https://arxiv.org/abs/2007.07447